A Necessary and Sufficient Condition for Stability and Harmonic Oscillations in Certain Second Order Differential Equation

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ABSTRACT

The issue of stability and oscillations existing in a second order differential equation has been examined in this paper. This is more general with equations of simple harmonic motions. By constructing a Liapunov function for the equation and using appropriate Liapunov theorems, some qualitative properties of the equation have been identified under various assumptions on the stiffness constant (k); namely stability and oscillations, stability, and instability.

(Keywords: Liapunov functions, stability, oscillation, solution paths, positive definite, positive semi-definite, and quadratic forms)

1.0 INTRODUCTION

Consider the second order differential equation:

\[ \ddot{x} + \frac{k}{m} x = 0 \]  

(1.1)

where \( k \) is a constant, or the equivalent system:

\[ \begin{align*}
\dot{x_1} &= x_2 \\
\dot{x_2} &= -\frac{k}{m} x_1 
\end{align*} \]  

(1.2)

Our concern is what happens to the solutions when \( k > 0, \ k = 0 \), and \( k < 0 \)?

By constructing Liapunov functions for the system (1.2) and using appropriate Liapunov theorems on stability and instability, the answers to these questions are readily available.

2.0 CONSTRUCTION OF A LIAPUNOV FUNCTION FOR EQUATION (1.1)

The construction of a Liapunov function for equation (1.1) shall be done in a close parallel with the method used by M. L. Cartwright in 1956. The procedure is to construct a Liapunov function for a constant coefficient equation:

\[ \dot{x} + ax + bx = 0 \]  

(2.1)

and then modify it to suit equation (1.1). Note that the equation (2.1) is equivalent to the system:

\[ \begin{align*}
\dot{x_1} &= x_2 \\
\dot{x_2} &= -ax_2 - bx_1 
\end{align*} \]  

(2.2)

The system (2.2) can be written compactly as:

\[ \dot{X} = AX \]  

(2.3)

where,

\[ A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]

The Eigenvalue equation for the matrix \( A \) is:

\[ \lambda^2 + a\lambda + b = 0 \]  

(2.4)

The roots of equation (2.4) have negative real parts, if, and only if (Ezeilo 1960):

\[ a > 0, \ b > 0 \]  

(2.5)
By choosing the most general quadratic form of order two and then picking the coefficients in the quadratic form to satisfy

\[ \dot{V} = -u \]  

(2.6)

along the solution paths of (2.2). Now consider \( V \) defined by \( V(x_1, x_2) \)

i.e. \( V = \alpha_1 x_1^2 + \alpha_2 x_2^2 + 2\alpha_3 x_1 x_2 \)  

(2.7)

where \( \alpha_1, \alpha_2, \alpha_3 \) are constants. Then along the solution paths of (2.2),

\[ \dot{V} = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_1 x_2 + \alpha_3 x_1 x_2 \]

\[ = \alpha_1 x_1^2 - \alpha_2 a x_2 - \alpha_3 b x_1 x_2 \]

\[ + \alpha_3 x_1^2 - \alpha_3 b x_1^2 - \alpha_3 a x_1 x_2 \]

Substituting the values of \( \alpha_1, \alpha_2, \alpha_3 \) into (2.7), we have the following:

\[ 2V = bx_1^2 + \left( ax_1^2 + x_2^2 + 2ax_1 x_2 \right) \]

\[ = bx_1^2 + \left( x_2 + ax_1 \right)^2 \]

which is positive definite subject to (2.5). We have therefore found positive definite quadratic form \( V \) given by (2.9) such that,

\[ \dot{V} = -abx_1^2 \]

corresponding to \( u = abx_1^2 \) which is positive semi definite by (2.5), (Ezeilo 1960).

Next we shall discuss how to construct a suitable Liapunov function for the system (1.2). Note that (1.2) is comparable with (2.2). That is (2.2) is the same as (1.2) if:

\[ \begin{align*}
  k & \text{ is replaced with } b, \\
  m & \text{ is replaced with } a \\
  a & \text{ is replaced by zero}
\end{align*} \]

(2.10)

The correlation between (1.2) and (2.2) described by (2.10) suggests that we consider a trial Liapunov function for (1.2) defined by:

\[ 2V = \frac{k}{m} x_1^2 + x_2^2 \]

(2.11)

since \( a \) does not appear explicitly in the system (1.2). Note that our trial \( V \) defined by (2.11) is positive semi definite if \( k > 0 \). The time derivative \( \dot{V} \) along the solution paths of (1.2) is zero. Therefore (2.11) is a suitable Liapunov function for the system (1.2).

### 3.0 DISCUSSION

Our discussion would be based on the following:

Consider the scalar equation

\[ \dot{x} = f(x), \quad f(0) = 0 \]  

(3.1)

\( f \) is sufficiently smooth, \( x \in \mathbb{R}^n \)

Theorem 1 (Liapunov first theorem)
Assumptions:

i. Let $f' \in c'$

ii. There exists a $c'$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) > 0$ for all $x$ and $V(x) = 0$ if $x = 0$.

Along the solution paths of (3.1) $\dot{V} \leq 0$ i.e. $f(x) \nabla V \leq 0$. Then the solution $x = 0$ of (3.1) is stable in the sense of Liapunov.

Theorem 2 (Liapunov)

Assumptions:

i. Let $f' \in c'$

ii. There exists a $c'$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that in every neighbourhood of the origin there is at least a point $x_0$ such that $V(x_0) > 0$.

iii. The time derivation $\dot{V}$ of $V$ along the solution paths of (3.1) is positive semi-definite i.e. $\dot{V} \geq 0$.

iv. The only solution $x(t)$ of (3.1) which satisfies $\dot{V}(x(t)) = 0$ is the trivial solution $x = 0$. Then the trivial solution $x \equiv 0$ of (3.1) is unstable.

(i) The $V$ defined by (2.11) i.e.

$$2V = k\frac{x_1^2 + x_2^2}{m}$$

is clearly positive semi-definite if $k > 0$ and the time derivative $\dot{V}$ along the solution paths of (1.2) is

$$\frac{k}{m}x_1x_2 - \frac{k}{m}x_1x_2 = 0.$$ 

Therefore our $V$ defined by (2.11) satisfies the thesis of theorem 1. Thus the system (1.2) is stable in the sense of Liapunov. Note also that the solution of equation (1.1) is harmonic. Thus for $k > 0$, the system (1.2) is stable and harmonic. That is both stability and harmonic oscillation.

(ii) The $V$ defined by (2.11) i.e.

$$2V = k\frac{x_1^2 + x_2^2}{m}$$

is positive semi definite if $k = 0$. The time derivative along the solution paths of (1.2) is $\dot{V} = \frac{k}{m}x_1x_2$ which is zero when $k = 0$. Therefore, for $k = 0$, the thesis of theorem 1 is also satisfied however the system (1.2) is only stable and no more harmonic.

(iii) Our $V$ defined by (2.1) can not be said to be positive definite or positive semi – definite if $k < 0$. Therefore, system (1.2) is neither stable nor harmonic.

CONCLUSION

There is stability and harmonic oscillations in the system (1.2) when the stiffness constant $k$ is greater than zero. Therefore a necessary and sufficient condition for stability and harmonic oscillations is that $k > 0$.

REFERENCES


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