Existence of Periodic Solutions for a certain Non-Linear Third Order Differential Equation

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ABSTRACT

In this paper, existence of periodic solutions for equations:

\[ \ddot{x} + f(\dot{x}) + g(t, \dot{x}) + a_1 x = p(t, x, \dot{x}, \ddot{x}) \quad \text{and} \]
\[ D^{(r)} x(0) = D^{(r)} x(2\pi), r = 0, 1, 2, \quad D = \frac{d}{dt} \]

has been obtained, when \( f(\dot{x}) \) is arbitrary, by the use of the Leray Schauder fixed-point technique and the integrated and so-called energy equation as the mode for estimating the a priori bounds.

(Keywords: fixed point technique, a priori bounds, compact and equicontinuity, integrated and so called energy equation)

1.0 INTRODUCTION

Consider the third order differential equation:

\[ \ddot{x} + f(\dot{x}) + g(t, \dot{x}) + a_1 x = p(t, x, \dot{x}, \ddot{x}) \quad \text{(1.1)} \]

with boundary conditions

\[ D^{(r)} x(0) = D^{(r)} x(2\pi), r = 0, 1, 2, \quad D = \frac{d}{dt} \quad \text{(1.2)} \]

where \( a_3 > 0 \) is a constant and \( f, g, \) and \( p \) are continuous functions depending on the arguments shown with \( p \) periodic in \( t \) that is

\[ P(t, x, \dot{x}, \ddot{x}) = p(t + 2\pi, x, \dot{x}, \ddot{x}) \]

In the special case:

\[ \dddot{x} + a\ddot{x} + b\dot{x} + cx = p(t) \quad \text{(1.3)} \]

in which \( a, b, c \) are constants and \( p \) is a continuous function and \( 2\pi \) periodic in \( t \). It is well known that if the Routh Hurwitz conditions hold,

\[ a > 0, \quad b > 0, \quad ab > c > 0 \quad \text{(1.4)} \]

the roots of the auxiliary equation

\[ \lambda^3 + a\lambda^2 + b\lambda + c = 0 \quad \text{(1.5)} \]

have negative real parts, so that the existence of periodic solutions when \( p \) is also \( 2\pi \) periodic in \( t \) can be verified for (1.3) when (1.4) holds.

In the literature, generalizations of these results are available for several non-linear third order differential equations. For instance, Ezeilo (1960) has proven the existence of at least one harmonic oscillation for the equation:

\[ \dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t) \quad \text{(1.6)} \]

where, \( h \) and \( p \) are continuous functions.

A similar result has also been proven by Pliss (1961) where the forcing term \( p \) now depends on \( t, x, \dot{x} \) and \( \ddot{x} \).

Villari (1964), proved the existence of periodic solutions for the equation:

\[ \dddot{x} + \varphi(\dot{x}) + b\dot{x} + c, x = p(t), b, < 0, c, > 0 \quad \text{(1.7)} \]
Subject to the condition:

\[ \{ \varphi(z_1) - \varphi(z_2) \} (z_1 - z_2) < 0 \quad \text{for} \quad z_1 \neq z_2 \quad (1.8) \]

However, condition (1.8) did not permit the use of the Leray–Schauder fixed-point technique.

Reissig, Sansone, and Conti (1974), considered the equation,

\[ \ddot{x} + \varphi(\dot{x}) + bx + cx = p(t) \quad \text{---------- (1.9)} \]

where \( b > 0, c > 0, |p(t)| \leq m \) for \( t > 0 \) or the corresponding system,

\[ \dot{x} = y, \dot{y} = z, z = p(t) - cx - by - \varphi(z) \quad (1.10) \]

Ezeilo (1986) proved the existence of periodic solutions for the equation

\[ \ddot{x} + g_4(\ddot{x}) + b_4 \dot{x} + c_4 x = p_4(t, x, \ddot{x}, \dot{x}) \quad (1.11) \]

where \( b_4, c_4 \) are positive constants and \( g_4, p_4 \) are continuous functions with,

\[ p(t, x, \dot{x}, \ddot{x}) = p(t + 2\pi, x, \dot{x}, \ddot{x}) \]

Ezeilo and Nkashama (1988), proved the existence of periodic solutions at resonance for the equation:

\[ \ddot{x} + a\ddot{x} + bx + g(t, x) = p(t, x, \ddot{x}, \dot{x}) \quad -- (1.12) \]

Anders (1985) proved the existence of 2\( \pi \) periodic solutions for the equation:

\[ \ddot{x} + f(t, x) + bx + cx = p(t) \quad \text{---------- (1.13)} \]

by constructing the Green’s function explicitly and weakening the growth conditions

\[ \lim_{|t| \to \infty} \frac{f(t, z)}{z} = 0 \]

by a concrete linear restriction on \( f \). When (1.4) is not fulfilled, the existence of 2\( \pi \) periodic solutions can still be established for a variety of equations (1.3) and generalization to nonlinear terms are known. Some examples are found in Reissig, Sansone, and Conti (1974).

For the more general cases like (1.1) – (1.2), it is difficult to locate in the literature or otherwise construct a suitable Lyapunov function corresponding to \( p = 0 \) which might be utilized in estimating \( \int_{0}^{2\pi} x^2 dt \) or \( \int_{0}^{2\pi} x^2 dt \).

The objective of this paper is to give some other result in the “non-Routh Hurwitz” direction. To be more precise, let us take the auxiliary equation (1.5) which has no purely imaginary root,

\[ \lambda = 2\pi \omega^{-1} \quad (\omega = 0), \quad \text{if} \quad ac > 0 \quad \text{and} \quad a^{-1}c \neq 4\pi^2 \omega^{-2}, \quad b \text{ arbitrary} \quad \text{---------- (1.14)} \]

Thus if \( p \) is 2\( \pi \) periodic in \( t \), the linear equation (1.3) has indeed a 2\( \pi \) periodic solution if \( a, b, c \) are also subject to (1.14) and we shall see here a suitable extension to equation (1.1). The results are summarized in the following:

**Theorem 1:** In addition to the basic assumptions on \( f, g \) and \( p \), suppose that

(i) There exists \( a_2 > 0 \) \( \alpha, \beta \) constant such that:

\[ \alpha + \beta = -1 \quad \text{---------- (1.15)} \]

(ii) \( g \) is a \( c^1 \) function such that:

\[ |g_x(t, \dot{x})| \leq \alpha, \quad |g_x(t, \ddot{x})| \leq \beta \quad \text{---------- (1.16)} \]

Then equations (1.1) – (1.2) have at least one 2\( \pi \) periodic solution for arbitrary \( f \).

**Remark:** The major interest is on \( g(t, \dot{x}) \) and \( f \). They have been used without any restrictions placed on them. It is also interesting to note that \( p \) is not subjected to further conditions other than the assumed continuity condition and 2\( \pi \) periodicity in \( t \).
2.0 NOTATION

The proof which follows denotes capitals $D_{0}, D_{1}, D_{2}, \cdots$ which depend on $f, g, a_{3}$ and $p$.

$D_{i,j}$ ($i=0,1,2, \cdots$) retains a fixed identity throughout the proof of Theorem 1. The symbols $[0,2\pi]$, $[1,2\pi]$, with respect to the mapping: $[0,2\pi] \rightarrow R$, will have their usual meaning.

That is for a given function $\theta : [0,2\pi] \rightarrow \square$;

\[
|\theta|_{\infty} \triangleq \max_{0 \leq t \leq 2\pi} |\theta(t)|, \quad |\theta| \triangleq \int_{0}^{2\pi} |\theta(s)| ds
\]

\[|\theta|_{2} \triangleq \left( \int_{0}^{2\pi} \theta^{2}(s) ds \right)^{1/2} \quad \text{(2.1)}
\]

3.0 PROOF OF THEOREM 1

The proof of theorem 1 is by the Leray Sechaunder fixed point technique and the starting point is the parameter $\lambda$ dependent equation:

\[
\ddot{x} + (1-\lambda) a_{2} \dot{x} + \lambda f(\dot{x}) + \lambda g(t, \dot{x}) + a_{3} x = \lambda p(t, x, \dot{x}, \ddot{x})
\]

or,

\[
\ddot{x} + f_{\dot{a}}(\dot{x}) + \lambda g(t, \dot{x}) + a_{3} x = \lambda p(t, x, \dot{x}, \ddot{x})
\]

\[\text{where}\]

\[
f_{\dot{a}}(\dot{x}) = (1-\lambda) a_{2} \dot{x} + \lambda f(\dot{x})
\]

by setting

\[
x = y, \dot{y} = z, \ddot{z} = -f_{\dot{a}}(\dot{x}) - \lambda g(t, \dot{x}) - a_{3} x + \lambda p
\]

\[\text{the equation (3.1) can be written compactly in matrix form as:}\]

\[
\dot{X} = AX + \lambda F(t, X)
\]

\[\text{where}\]

\[
X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_{3} & 0 & -a_{2} \end{bmatrix}, F = \begin{bmatrix} 0 \\ Q \end{bmatrix}
\]

where

\[
Q = -f(\ddot{x}) + a_{2} \ddot{x} - g(t, \dot{x}) + \lambda p
\]

We remark that equation (3.1) reduces to a linear equation:

\[
\ddot{x} + a_{2} \ddot{x} + a_{3} x = 0 \quad \text{---------------------- (3.6)}
\]

when $\lambda = 0$ and to equation (1.1) when $\lambda = 1$.

The eigenvalues of $A$ can be verified to be the roots of the auxiliary equation of (3.6), namely

\[
r^{3} + a_{3} r^{2} + a_{3} = 0 \quad \text{---------------------- (3.7)}
\]

By (1.14) and with $a_{3} > 0$, the equation (3.7) has no roots of the form $r = i\beta$, ($\beta$ real). Thus, the matrix $(\ell^{-2\pi A} - I)$, $I$ (being the identity $3 \times 3$ matrix) is invertible. Therefore $X = X(t)$ is a $2\pi$ periodic solution of (3.4) if and only if $X$ satisfies the equation:

\[
X = \lambda TX \quad \text{---------------------- (3.8)}
\]

Where the transformation $T$ is defined by:

\[
(tX)(t) = \int_{t}^{t+2\pi} \ell^{-2\pi A} - I \ell^{(t-s)A} F(s, X(s)) ds
\]

\[\text{where}\]

\[
\ell^{-2\pi A} - I \ell^{(t-s)A} F(s, X(s)) ds
\]

Let $S$ be the space of all real continuous and
3 – vector function \( \vec{X}(t) = (\vec{x}(t), \vec{y}(t), \vec{z}(t)) \)

which are of period \( 2\pi \) and with norm \( \|\vec{X}\| \) of \( \vec{X}(t) = (\vec{x}(t), \vec{y}(t), \vec{z}(t)) \) defined by:

\[
\|\vec{X}\| = \sup_{0 \leq t \leq 2\pi} \left\{ |\vec{x}(t)| + |\vec{y}(t)| + |\vec{z}(t)| \right\} \tag{3.10}
\]

If the mapping of \( T \) is completely continuous, mapping of into itself it suffices for the proof of theorem: merely to establish apriori bounds. Next is to show that the conditions of Schaefer’s lemma [1955] are satisfied under the hypotheses of Theorem 1. This requires the proof of the following:

**Lemma 1.**

Let \( T \) be a compact transformation of a normed linear space \( S \) into itself. Let \( \lambda \in [0,1] \). Then either there is an \( x \in S \) such that \( x = \lambda \cdot T(x) \) or the set \( \{x \in S : x = \lambda \cdot T(x), \ 0 < \lambda < 1\} \) is not bounded. For further details on the proof of Lemma 1, see Tejumola 1966.

Finally the proof of Theorem 1 will suffice to concentrate on equation (1.3.1) and to prove simply that there exists a constant \( D_0 > 0 \) independent of \( \lambda \) such that

\[
|x|_e \leq D_0, |x|_e \leq D_0, |\dot{x}|_e \leq D_0, \quad \ldots \ldots \tag{3.11}
\]

**4.0 VERIFICATION OF THE APRIORI BOUNDS**

Let \( x(t) \) be a possible \( 2\pi \) periodic solution of \( (3.1) \). The main tool to be used in this verification is the integrated and so called energy equation \( \dot{W} \) defined by:

\[
\dot{W} = -\ddot{x}^2 + \dddot{x}g_x(t, x) + \dddot{x}g_x(t, x) + \lambda p\dddot{x} \tag{4.2}
\]

Integrating (4.2) with respect to \( t \) from \( t = 0 \) to \( t = 2\pi \) and using the \( 2\pi \) periodicity condition, we have the following:

\[
0 \leq -\int_0^{2\pi} \dddot{x}^2 dt + \int_0^{2\pi} \dddot{x}g_x(t, x) dt + \int_0^{2\pi} \dddot{x}g_x(t, x) dt + \int_0^{2\pi} p\dddot{x} dt
\]

By equation (1.15), we have:

\[
\int_0^{2\pi} \dddot{x}^2 dt - \alpha \int_0^{2\pi} \dddot{x}^2 dt - \beta \int_0^{2\pi} \dddot{x}^2 dt \leq |p| \int_0^{2\pi} |\dddot{x}| dt
\]

In particular,

\[
\int_0^{2\pi} \dddot{x}^2 dt \leq D_1 \int_0^{2\pi} |\dddot{x}| dt \leq D_1 (2\pi)^{1/2} \left( \int_0^{2\pi} \dddot{x}^2 dt \right)^{1/2}
\]

by Schwartz’s inequality.

Therefore:

\[
\left( \int_0^{2\pi} \dddot{x}^2 dt \right)^{1/2} \leq D_1 (2\pi)^{1/2} = D_2 \quad \ldots \ldots \tag{4.3}
\]

Now since \( \dddot{x}(0) = \dddot{x}(2\pi) \) implies that there exists \( \dddot{x}(\tau) = 0 \) for some \( \tau \in [0, 2\pi] \), such that:

\[
\dddot{x}(t) = \dddot{x}(\tau) + \int_{\tau}^t \dddot{x}(s) ds
\]

Therefore, by Schwartz’s inequality, the following can be derived from equation (4.3):

\[
\max_{0 \leq t \leq 2\pi} |\dddot{x}(t)| \leq (2\pi)^{1/2} D_2 \equiv D_3
\]

Thus,

\[
|\dddot{x}|_e \leq D_3 \quad \ldots \ldots \tag{4.4}
\]
The periodicity condition $x(0) = x(2\pi)$ on $x(t)$ implies that there exists the following:

$\tau \in [0,2\pi]$ such that $\dot{x}(\tau) = 0$. Thus the identity $\dot{x}(t) = \dot{x}(\tau) + \int_\tau^t \ddot{x}(s) \, ds$ holds.

That is, $\dot{x}(t) = \int_\tau^t \ddot{x}(s) \, ds$

Therefore, by equation (4.4),

$$\text{Max}_{0 \leq \tau \leq 2\pi} |\dot{x}(t)| \leq (2\pi)^{\frac{1}{2}} D_\tau \equiv D_4.$$ 

Thus,

$$|\dot{x}|_x \leq D_4 \quad \text{(4.5)}$$

All that remains now is for the first inequality for (3.11) to be fully established.

By integrating (3.1) with respect to $t$ from $t = 0$ to $t = 2\pi$ and using the $2\pi$ periodicity condition:

$$\int_0^{2\pi} f_2(\tilde{x}) \, dt + \int_0^{2\pi} \lambda g(t,\tilde{x}) \, dt + \int_0^{2\pi} a_3 x dt = \int_0^{2\pi} \lambda p dt$$

That is,

$$\int_0^{2\pi} a_3 x dt = \int_0^{2\pi} p dt - \int_0^{2\pi} f(\tilde{x}) \, dt - \int_0^{2\pi} g(t,\tilde{x}) \, dt$$

since $p$ is continuous and $2\pi$ periodic in $t$, $p$ is definitely bounded. Similarly,

$$\int_0^{2\pi} f(\tilde{x}) \, dt \quad \text{and} \quad \int_0^{2\pi} g(t,\tilde{x}) \, dt$$

in view of (4.4) and (4.5) $f(\tilde{x})$ and $g(t,\tilde{x})$ are bounded.

Thus,

$$\int_0^{2\pi} a_3 x dt \leq D_5$$

In particular,

$$\int_0^{2\pi} x dt \leq D_6 \quad \text{(4.6)}$$

Now let

$$|x(\tau)| \leq D_7 \quad \text{(4.7)}$$

for some $\tau \in [0,2\pi]$. Thus the identity

$$x(t) = x(\tau) + \int_\tau^t \ddot{x}(s) \, ds$$

holds.

Suppose not, that is:

$$\int_0^{2\pi} x dt \geq D_7 \quad \text{(4.8)}$$

then equation (4.6) is violated. Therefore equation (4.6) holds. Thus,

$$\text{Max}_{0 \leq \tau \leq 2\pi} |x(t)| \leq \frac{1}{2} \left( \int_0^{2\pi} \dot{x}^2(t) \, dt \right)^{\frac{1}{2}}$$

by Schwartz's inequality. From (4.5),

$$|\dot{x}|_x \leq D_8 \quad \text{(4.9)}$$

The estimates (4.4),(4.5) and (4.9) verify (3.11) and thus follows the existence of $2\pi$ periodic solutions for equation (1.1)–(1.2).

REFERENCES


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SUGGESTED CITATION


The Pacific Journal of Science and Technology
http://www.akamaiusiversity.us/PJST.htm

Volume 7. Number 2. November 2006 (Fall)