An Efficient Hybrid Algorithm for the Computation of Second-Order Fredholm Integro-Differential Equations

J. Sunday[1*]; Y. Skwame[2]; O. Sarjiyus[3]; I. Manga[3]; J. A. Mohammed[3] and B.J. Neils[3]

[1]Department of Mathematics, University of Jos, Nigeria
[2]Department of Mathematics, Adamawa State University, Mubi, Nigeria.
[3]Department of Computer Science, Adamawa State University, Mubi, Nigeria.

E-mail: joshuasunday2000@yahoo.com*
sundayjo@unijis.edu*

ABSTRACT

In this paper, an efficient hybrid algorithm shall be formulated for the computation of second-order Fredholm integro-differential equations. In developing the algorithm using the method of interpolation and collocation, power series was adopted as the basis function with the integration carried out within a one-step interval. The algorithm derived was then applied on some modeled second-order Fredholm integro-differential equations and from the results obtained; it is obvious that the algorithm is computationally reliable. The basic properties of the algorithm derived were also analyzed.

(Keywords: algorithm, Fredholm equations, hybrid integro-differential equations, second-order)
(AMS Subject Classification: 65L05, 65L06, 65D30)

INTRODUCTION

The integro-differential equation is one of the most applied equations in science and engineering. It was introduced by Volterra for the first time in the early 1900s. It is an equation that involves both integrals and derivatives of a function. Volterra investigated the population growth, focusing his study on the hereditary influences; where through his research work the topic of integro-differential equations was established by Wazwaz (2011).

It is important to note that in the integro-differential equations, the unknown function $y(x)$ and one or more of its derivatives such as $y'(x), y''(x),...$ appear out and under the integral sign as well (Mohammed *et. al.*, 2016). It can be classified into Fredholm equations and Volterra equations. The upper bound of the region for integral part of Volterra type is a variable while it is a fixed number for that of Fredholm type.

In this paper, a highly efficient hybrid algorithm shall be developed for the computation of second order Fredholm integro-differential equations of the form:

$$y''(x) = f(x,y) + \int_a^b k(x,t)y(t)dt, \quad a \le x \le b \qquad \textbf{(1)}$$

subject to the initial conditions:

$$y(a) = \alpha, \quad y'(a) = \beta \qquad \textbf{(2)}$$

where α *and* β are real constants. The function $f(x,y)$ and the kernel $k(x,t)$ are known. The solution $y(x)$ is to be determined. We assume that the problem (1) is well-posed; that is, the problem has the following properties:

- a solution exists,
- the solution is unique, and
- the solution's behavior changes continuously with the initial conditions

Fredholm integro-differential equations model many situations in science and engineering, such as in circuit analysis. The activity of interacting inhibitory and excitatory neurons can be described by a system of integro-differential equations. They are also of significant importance in modeling numerous physical processes such as signal processing and neural networks (Kanwal, 1997). The applications of Fredholm integro-differential equations in electromagnetic theory and dispersive waves and ocean circulations are enormous, Mohammed *et al.* (2016).

Many authors have developed different methods for the solution of problems of the form (1). These methods include Lagrange interpolation method (Rashed, 2004), Tau operational method (Mohammad and Shahmorad, 2005), Legendre polynomial method (Saadatmandi and Dehghan, 2010), generalized minimal residual method (Aruchuman and Sulaiman, 2010), differential transform method with Adomian polynomials (Behiry, 2013), canonical basis function method (Taiwo, Ganiyu and Okperhie, 2014), power series and Chebyshev series approximation methods (Gegele, Evans, and Akoh, 2014), Bessel function method (Parand and Nikarya, 2014), cubic spline collocation method (Taiwo and Gegele, 2014), nonstandard finite difference method (Pandey, 2015), homotopy analysis transform method (Mohammed *et al.,* 2016), among others.

However, in this research we shall develop an efficient hybrid algorithm and apply it to compute Fredholm integro-differential equations of the form (1). It is important to state that the hybrid algorithm has the advantage of generating independent solutions at selected grid points without overlapping. It is also less expensive in terms of the number of function evaluation compared to predictor-corrector methods; moreover, they possess the properties of Runge-Kutta method of being self-starting and do not require starting values.

FORMULATION OF THE HYBRID ALGORITHM

In this section, a discrete hybrid algorithm of the form:

$$A^{(0)}\mathbf{Y}_m^{(i)} = \sum_{i=0}^{1} h^i e_i y_n^{(i)} + h^2 d_i f(y_n) + h^2 b_i f(\mathbf{Y}_m),\ i = 0,1 \qquad \textbf{(3)}$$

shall be derived for the computation of Fredholm integro-differential equations of the form (1) on the interval $[x_n, x_{n+1}]$. The initial assumption is that the solution on the interval $[x_n, x_{n+1}]$ is locally approximated by the basis function (approximate solution),

$$y(x) = \sum_{j=0}^{r+s-1} \tau_j x^j \qquad \textbf{(4)}$$

where τ_j are the real coefficients to be determined, r is the number of interpolation points, s is the number of collocation points and $h = x_n - x_{n-1}$ is a constant step-size of the partition of the interval $[\alpha, \beta]$ which is given by $\alpha = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = \beta$.

We assume that the polynomial (4) must pass through the interpolation points $(x_{n+s}, y_{n+s}), s = \frac{1}{2}, \frac{3}{4}$ and the interpolation points $(x_{n+r}, f_{n+r}), r = 0\left(\frac{1}{4}\right)1$ and we require that the following $(r+s)$ equations must be satisfied:

$$\sum_{j=0}^{r+s-1} \tau_j x^j = y_{n+s},\ s = \frac{1}{2}, \frac{3}{4} \qquad \textbf{(5)}$$

$$\sum_{j=0}^{r+s-1} j(j-1)\tau_j x^{j-2} = f_{n+r},\ r = 0\left(\frac{1}{4}\right)1 \qquad \textbf{(6)}$$

The $(r+s)$ undetermined coefficients τ_j are obtained by solving the system of nonlinear equations (5) and (6) using Gauss elimination method. This gives a continuous hybrid linear multistep algorithm of the form:

$$y(x) = \alpha_{\frac{1}{2}}(t) y_{n+\frac{1}{2}} + \alpha_{\frac{3}{4}}(t) y_{n+\frac{3}{4}} + h^2 \left(\sum_{j=0}^{1} \beta_j(t) f_{n+j} + \beta_k(t) f_{n+k} \right),\ k = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \qquad \textbf{(7)}$$

The coefficients $\alpha_{\frac{1}{2}}, \alpha_{\frac{3}{4}}, \beta_0, \beta_{\frac{1}{4}}, \beta_{\frac{1}{2}}, \beta_{\frac{3}{4}}, \beta_1$ are given by;

$$\left.\begin{aligned}
&\alpha_{\frac{1}{2}}(t) = 3 - 4t \\
&\alpha_{\frac{3}{4}}(t) = 4t - 2 \\
&\beta_0(t) = \frac{1}{11520}(4096t^6 - 15360t^5 + 22400t^4 - 16000t^3 + 5760t^2 - 950t + 51) \\
&\beta_{\frac{1}{4}}(t) = -\frac{1}{2880}(4096t^6 - 13824t^5 + 16640t^4 - 7680t^3 + 954t - 189) \\
&\beta_{\frac{1}{2}}(t) = \frac{1}{1920}(4096t^6 - 12288t^5 + 12160t^4 - 3840t^3 - 322t + 201) \\
&\beta_{\frac{3}{4}}(t) = -\frac{1}{2880}(4096t^6 - 10752t^5 + 8960t^4 - 2560t^3 + 142t - 39) \\
&\beta_1(t) = \frac{1}{11520}(4096t^6 - 9216t^5 + 7040t^4 - 1920t^3 + 66t - 9)
\end{aligned}\right\} \quad \textbf{(8)}$$

where $t = \dfrac{x - x_n}{h}$, $y_{n+j} = y(t_n + jh)$ is the numerical approximation to the analytic solution $y(t_{n+j})$ and $y_{n+j} = f_{n+j} = f\left((t_n + jh), y(t_n + jh), y'(t_n + jh)\right)$ is the approximation to $y'(t_{n+j})$.

The continuous algorithm (7) is then solved for the independent solution at the grid points to give the continuous algorithm:

$$y(t) = \sum_{j=0}^{1} \frac{(jh)^{(m)}}{m!} y_n^{(m)} + h^2 \left(\sum_{j=0}^{1} \sigma_j(t) f_{n+j} + \sigma_k f_{n+k} \right), \; k = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \quad \textbf{(9)}$$

The coefficients $\sigma_i,\ i = 0\left(\frac{1}{4}\right)1$ gives:

$$\left.\begin{aligned}
&\sigma_0(t) = \frac{1}{90}\left(32t^6 - 120t^5 + 175t^4 - 125t^3 + 45t^2\right) \\
&\sigma_{\frac{1}{4}}(t) = -\frac{1}{45}\left(64t^6 - 216t^5 + 260t^4 - 120t^3\right) \\
&\sigma_{\frac{1}{2}}(t) = \frac{1}{15}\left(32t^6 - 96t^5 + 95t^4 - 30t^3\right) \\
&\sigma_{\frac{3}{4}}(t) = -\frac{1}{45}\left(64t^6 - 168t^5 + 140t^4 - 40t^3\right) \\
&\sigma_1(t) = \frac{1}{90}\left(32t^6 - 72t^5 + 55t^4 - 15t^3\right)
\end{aligned}\right\} \quad \textbf{(10)}$$

We then evaluate (9) at $t = \frac{1}{4}\left(\frac{1}{4}\right)1$ to give the algorithm of the form (3) where,

$$\mathbf{Y}_m = \begin{bmatrix} y_{n+\frac{1}{4}} & y_{n+\frac{1}{2}} & y_{n+\frac{3}{4}} & y_{n+1} \end{bmatrix}^T,$$

$$\mathbf{y}_n^{(i)} = \begin{bmatrix} y^{(i)}_{n-\frac{1}{4}} & y^{(i)}_{n-\frac{1}{2}} & y^{(i)}_{n-\frac{3}{4}} & y^{(i)}_n \end{bmatrix}^T$$

$$\mathbf{F}(\mathbf{Y}_m) = \begin{bmatrix} f_{n+\frac{1}{4}} & f_{n+\frac{1}{2}} & f_{n+\frac{3}{4}} & f_{n+1} \end{bmatrix}^T,$$

$$\mathbf{f}(\mathbf{y}_n) = \begin{bmatrix} f_{n-\frac{1}{4}} & f_{n-\frac{1}{2}} & f_{n-\frac{3}{4}} & f_n \end{bmatrix}^T$$

$A^{(0)}$ is a 4×4 identity matrix given by:

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When $i = 0$:

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad e_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad d_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{367}{23040} \\ 0 & 0 & 0 & \frac{53}{1440} \\ 0 & 0 & 0 & \frac{147}{2560} \\ 0 & 0 & 0 & \frac{7}{90} \end{bmatrix}$$

$$b_0 = \begin{bmatrix} \frac{3}{128} & \frac{-47}{3840} & \frac{29}{5760} & \frac{-7}{7680} \\ \frac{1}{10} & \frac{-1}{48} & \frac{1}{90} & \frac{-1}{480} \\ \frac{117}{640} & \frac{27}{1280} & \frac{3}{128} & \frac{-9}{2560} \\ \frac{4}{15} & \frac{1}{15} & \frac{4}{45} & 0 \end{bmatrix}$$

When $i=1$:

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad d_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{2880} \\ 0 & 0 & 0 & \frac{29}{360} \\ 0 & 0 & 0 & \frac{27}{320} \\ 0 & 0 & 0 & \frac{7}{90} \end{bmatrix} \quad b_1 = \begin{bmatrix} \frac{323}{1440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} \\ \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} \\ \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{320} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix}$$

ANALYSIS OF BASIC PROPERTIES OF THE HYBRID ALGORITHM

In this section, the basic properties of the hybrid algorithm shall be analyzed.

Order of Accuracy and Error Constant of the Hybrid Algorithm

According to Lambert (1991), the linear operator associated with the discrete hybrid algorithm (3) is defined as:

$$L\{y(t):h\} = \mathbf{A}^{(0)}\mathbf{Y}_m^{(i)} - \sum_{i-0}^{1} h^i e_i y_n^{(i)} - h^2\left(d_0 f(y_n) + b_0\mathbf{F}(\mathbf{Y}_m)\right) \qquad \textbf{(11)}$$

Assuming that $y(t)$ is sufficiently differentiable, we write the terms in (11) as a Taylor series expansion about the point t to obtain the expression:

$$L\{y(t):h\} = c_0 y(t) + c_1 hy'(t) + c_2 h^2 y''(t) + \ldots + c_p h^p y^p(t) + c_{p+1} h^{p+1} y^{p+1}(t) + c_{p+1} h^{p+2} y^{p+2}(t) \qquad \textbf{(12)}$$

where the constant coefficients $c_p, p = 0,1,2,\ldots$ are given by;

$$\left.\begin{aligned}
c_0 &= \sum_{j=0}^{k} \alpha_j \\
c_1 &= \sum_{j=0}^{k} \left(j\alpha_j - \beta_j \right) \\
&. \\
&. \\
&. \\
c_p &= \sum_{j=0}^{k} \left[\frac{1}{q!} j^q \alpha_j - \frac{1}{(q-1)!} j^{q-1} \beta_j \right], \; q = 2,3,\ldots
\end{aligned}\right\} \quad \textbf{(13)}$$

The hybrid algorithm (3) is said to be of uniform accurate order p , if p is the largest positive integer for which $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \ldots = \bar{c}_p = \bar{c}_{p+1} = 0,\ \bar{c}_{p+2} \neq \bar{0}$. $\bar{c}_{p+2}$ is called the error constant and the local truncation error of the algorithm is given by;

$$\bar{t}_{n+k} = \bar{c}_{p+2} h^{(p+2)} y^{(p+2)}(t) + O\left(h^{(p+3)}\right) \quad \textbf{(14)}$$

t has therefore been established from our computations that the hybrid algorithm (3) has coefficients of h given by $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \bar{c}_5 = \bar{c}_6 = \bar{0}$, implying that the order $p = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}^T$ and the error constant is given by:

$$\bar{c}_7 = \left[6.4789\times10^{-7} \;\; 1.5501\times10^{-6} \;\; 2.4523\times10^{-6} \;\; 3.1002\times10^{-6} \right]^T$$

Consistency of the Hybrid Algorithm

The hybrid algorithm (3) is consistent since it has order $p = 5 \geq 1$. According to Fatunla (1980), consistency controls the magnitude of the local truncation error committed at each stage of the computation.

Zero-Stability of the Hybrid Algorithm

Definition 1: (Fatunla, 1980): The hybrid algorithm (3) is said to be zero-stable, if the roots $z_{s,} s = 1,2,\ldots,k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(zA^{(0)} - e_0)$ satisfies $|z_s| \leq 1$ and every root satisfying $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation.

Moreover, as $h \to 0, \rho(z) = z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the matrices $A^{(0)}$ *and* e_0. For our method,

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0 \quad (15)$$

Therefore,

$$\rho(z) = j \to \infty \Rightarrow z_1 = z_2 = z_3 = 0,\ z_4 = 1.$$

Hence, the hybrid algorithm is zero-stable. It is important to note that the main consequence of zero-stability is to control the propagation of the error as the integration proceeds.

Convergence of the Hybrid Algorithm

The hybrid algorithm is convergent since it is consistent and zero-stable.

Theorem 1 (Butcher, 2008)

A linear multistep hybrid algorithm is convergent if and only if it is stable and consistent.

Region of Absolute Stability of the Hybrid Algorithm

Definition 2 (Yan, 2011)

Region of absolute stability is a region in the complex z plane, where $z=\lambda h$. It is defined as those values of z such that the numerical solutions of $y''=-\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

In determining the stability polynomial of the hybrid algorithm derived, the boundary locus method will be adopted. This gives:

$$\bar{h}(w) = -h^8\left(\frac{7}{3686400}w^3 - \frac{1}{4915200}w^4\right) + h^6\left(\frac{1}{1474560}w^4 - \frac{1123}{2211840}w^3\right) - h^4\left(\frac{307}{9216}w^3 - \frac{31}{92160}w^4\right) - h^2\left(\frac{5}{192}w^4 + \frac{59}{96}w^3\right) + w^4 - 2w^3 \quad \textbf{(16)}$$

The stability region is shown in Figure 1.

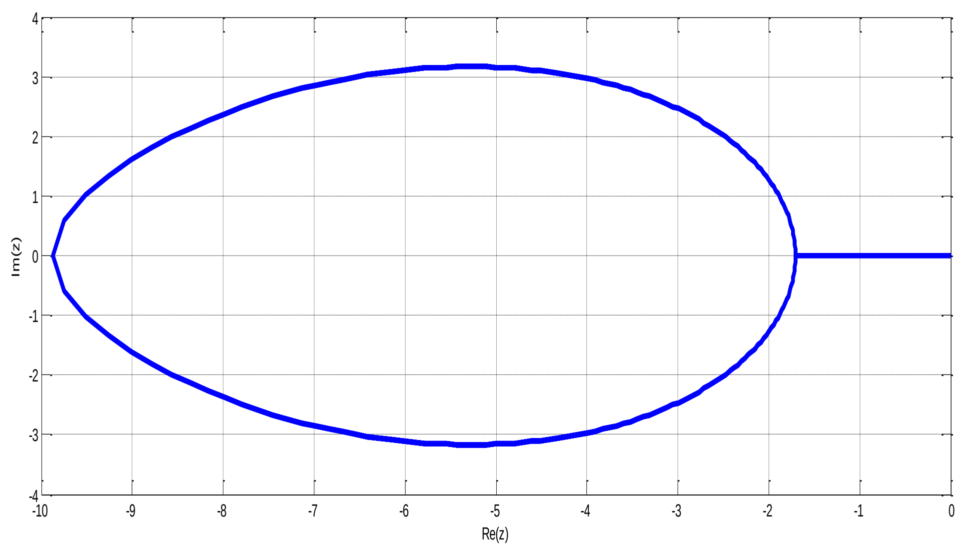

Figure 1: Region of the Absolute Stability of the Hybrid Algorithm.

The stability region in the Figure 1 is A-stable

RESULTS AND DISCUSSION

Numerical Experiments

The hybrid algorithm developed in this research shall be adopted in solving some modeled real-life Fredholm integro-differential equations of the form (1). The following notations shall be used in the tables below:

- AEHA - Absolute error of the algorithm
- EMGAA - Absolute error in Mohammed *et. al.* (2016)
- EGEA - Absolute error in Gegele *et. al*. (2014)
- ETG - Absolute error in Taiwo and Gegele (2014)
- $t/\sec$ - Execution time per seconds for computation at each stage

Problem 1: Consider the model Fredholm integro-differential equation:

$$y''(x) = e^x - x + \int_0^1 xty(t)dt, \quad 0 \le x \le 1 \tag{17}$$

subject to the initial conditions:

$$y(0) = 1, \;\; y'(0) = 1 \tag{18}$$

The exact solution to the problem is given by:

$$y(x) = e^x \tag{19}$$

Source: Mohammed *et. al.* (2016)

On the application of the new hybrid algorithm on Problem 1 we obtain the result presented in Table 1.

Table 1: Absolute Error of the Hybrid Algorithm for Problem 1.

x	AEHA	EGEA	$t/\sec$
0.1000	3.024248e-013	2.01e-008	0.0294
0.2000	4.584944e-013	1.27e-008	0.0396
0.3000	7.316370e-014	1.36e-007	0.0463
0.4000	1.692257e-012	5.25e-007	0.0568
0.5000	4.596878e-012	2.29e-006	0.0664
0.6000	8.754997e-012	3.98e-006	0.0667
0.7000	1.390665e-011	1.59e-005	0.0668
0.8000	1.959244e-011	7.76e-004	0.0670
0.9000	2.519718e-011	3.67e-004	0.0671
1.0000	2.999911e-011	5.65e-004	0.0672

Problem 2: Consider the model Fredholm integro-differential equation:

$$y''(x) = 32x + \int_{-1}^{1} (1 - xt)y(t)dt, \quad -1 \le x \le 1 \tag{20}$$

subject to the initial conditions:

$$y(0) = 1, \;\; y'(0) = 1 \tag{21}$$

The exact solution to the problem is given by:

$$y(x) = 1 + \frac{3}{2}x^2 + 5x^3 \tag{22}$$

Source: Gegele *et. al.* (2014)

On the application of the new hybrid algorithm on Problem 2 we obtain the result presented in Table 2.

Table 2: Absolute Error of the Hybrid Algorithm for Problem 2.

x	AEHA	EGEA	$t/\sec$
0.1000	0.000000e+000	1.250e-005	0.0136
0.2000	1.110223e-016	5.000e-004	0.0215
0.3000	8.881784e-016	4.375e-004	0.0218
0.4000	7.771561e-016	3.700e-004	0.0220
0.5000	4.440892e-016	2.625e-004	0.0223
0.6000	1.665335e-015	2.000e-004	0.0225
0.7000	2.775558e-015	1.875e-004	0.0226
0.8000	5.440093e-015	1.300e-003	0.0317
0.9000	7.216450e-015	1.212e-003	0.0319
1.0000	9.436896e-015	2.500e-003	0.0321

Problem 3: Consider the model Fredholm integro-differential equation:

$$y''(x) = \frac{5}{3} - 11x + \int_0^1 y(t)dt, \quad 0 \le x \le 1 \tag{23}$$

subject to the initial conditions:

$$y(0) = y'(0) = 1 \tag{24}$$

The exact solution to the problem is given by:

$$y(x) = 1 + x + \frac{5}{6}x^2 - \frac{5}{3}x^3 \tag{25}$$

Source: Taiwo and Gegele (2014)

On the application of the new hybrid algorithm on Problem 3 we obtain the result presented in Table 3.

Table 3: Absolute Error of the Hybrid Algorithm for Problem 3.

x	AEHA	ETG	$t/\sec$
0.1000	2.292055e-012	3.489e-006	0.0116
0.2000	3.113954e-012	3.410e-006	0.0231
0.3000	3.376410e-012	2.983e-006	0.0236
0.4000	3.424150e-012	2.837e-006	0.0237
0.5000	3.394396e-012	2.602e-006	0.0241
0.6000	3.343548e-012	2.591e-006	0.0242
0.7000	3.294920e-012	2.429e-006	0.0243
0.8000	3.257394e-012	1.994e-006	0.0244
0.9000	3.234413e-012	1.405e-006	0.0245
1.0000	3.226530e-012	1.067e-008	0.0247

Problem 4: Consider the model Fredholm integro-differential equation:

$$y''(x) = 10 - \frac{146}{35}x + \frac{1}{2}\int_{-1}^{1} xty^2(t)dt, \quad -1 \le x \le 1 \tag{26}$$

subject to the initial conditions:

$$y(0) = 1, \; y'(0) = 0 \tag{27}$$

The exact solution to the problem is given by:

$$y(x) = 1 + 5x^2 - x^3 \tag{28}$$

Source: Taiwo and Gegele (2014)

On the application of the new hybrid algorithm on Problem 4 we obtain the result presented in Table 4.

Table 4: Absolute Error of the Hybrid Algorithm for Problem 4.

x	AEHA	ETG	$t/\sec$
0.1000	1.147693e-014	6.008e-008	0.0124
0.2000	6.714074e-014	7.918e-008	0.0218
0.3000	1.834088e-013	8.432e-007	0.0312
0.4000	3.385625e-013	6.884e-007	0.0316
0.5000	4.861112e-013	5.718e-007	0.0320
0.6000	5.798695e-013	5.623e-007	0.0321
0.7000	5.947465e-013	4.009e-007	0.0322
0.8000	5.329071e-013	2.929e-007	0.0323
0.9000	4.160006e-013	2.887e-007	0.0325
1.0000	2.744471e-013	1.999e-007	0.0326

DISCUSSION OF RESULTS

From the results generated in Tables 1-4, it is clear that the newly derived hybrid algorithm is computationally reliable and efficient. This is because the absolute errors of the hybrid algorithm derived (AEA) is far smaller than those of the methods with which we compared our results with. The hybrid algorithm is also efficient because from the tables, the execution times per seconds are very small. This shows that the hybrid algorithm generates results very fast.

CONCLUSION

An efficient hybrid algorithm has been developed in this paper for the solution of second order Fredholm Integro-differential equations of the form (1). The hybrid algorithm developed was applied on some modeled second order Fredholm integro-differential equations and from the results obtained, it is clear that the hybrid algorithm is computationally reliable. The analysis of the hybrid algorithm derived was also carried out showing that it is consistent, convergent and zero-stable.

REFERENCES

1. Aruchuman, E. and J. Sulaiman. 2010. "Numerical Solution of Second Order Linear Fredholm Integro-Differential Equation Using Generalized Minimal Residual Method". *American Journal of Applied Sciences*. 7(6):780-783

2. Behiry, S.H. 2013. "Nonlinear Integro-Differential Equations by Differential Transform Method with Adomian Polynomials". *Mathematical Science Letters*. 2(3):209-221. Dx.doi.org/10.12785/msl/020310

3. Butcher, J.C. 2008. *Numerical Methods for Ordinary Differential Equations*. John Wiley and Sons Ltd, Chichester, England, 2nd Edition.

4. Fatunla, S.O. 1980. "Numerical Integrators for Stiff and Highly Oscillatory Differential Equations". *Mathematics of Computation*. 34:373-390.

5. Gegele, O.A., O.P. Evans, and D. Akoh. 2014. "Numerical Solution of Higher Order Linear Fredholm Integro-Differential Equations". *American Journal of Engineering Research*. 3(8):243-247.

6. Kanwal, R.P. 1997. *Linear Integral Equations*. Birkhauser: Boston, MA.

7. Lambert, J.D.1991. *Numerical Methods for Ordinary Differential Systems: The Initial Value Problems*. John Wiley and Sons LTD: London, UK.

8. Mohammad, S.H. and S. Shahmorad. 2005. "Numerical Piecewise Approximate Solution of Fredholm Integro-Differential Equations by the Tau Method". *Journal of Applied Mathematical Modeling*. 29:1005-1021.

9. Mohammed, M.S., K.A. Gepreel, M.R. Alharthi, and R.A. Alotabi. 2016. "Homotopy Analysis Transform Method for Integro-Differential Equations". *Gen. Maths. Notes*. 32(1):32-48.

10. Pandey, P.K. 2015. "Nonstandard Finite Difference Method for Numerical Solution of Second Order Linear Fredholm Integro-Differential Equations". *International Journal of Mathematical Modeling and Computations*. 5(3):259-266.

11. Parand, K. and M. Nikarya. 2014. "Application of Bessel Functions for Solving Differential and Integro-Differential Equations of the Fractional Order". *Applied Mathematical Modeling*. 38:4137-4147. Dx.doi.org/j.apm.2014.02.001.

12. Rashed, M.T. 2004. "Lagrange Interpolation to Compute Numerical Solution of Difference, Integral and Integro-Differential Equations". *Journal of Applied Mathematics and Computation*. 151:869-878.

13. Saadatmandi, A. and M. Dehghan. 2010. "Numerical Solution of the Higher-Order Linear Fredholm Integro-Differential-Difference Equation with Variable Coefficients". *Computer and Mathematics with Applications*. 59:2996-3004. DOI:10.1016/j.camwa.2010.02.018.

14. Taiwo, O.A., K.A. Ganiyu, and E.P. Okperhie. 2014. "Numerical Solution of Second Order Nonlinear Fredholm-Volterra Integro-Differential Equations by Canonical Basis Function". *International Journal of Engineering and Science*. 4(1):46-51.

15. Taiwo, O.A. and O.A. Gegele. 2014. "Numerical Solution of Second Order Linear and Nonlinear Integro-Differential Equations by Cubic Spline Collocation Method". *Advancement in Scientific and Engineering Research*. 2(2):18-22.

16. Wazwaz, A.M. 2011. *Linear and Nonlinear Integral Equations Methods and Applications*. Springer: New York, NY.

17. Yan, Y.L. 2011. *Numerical Methods for Differential Equations*. City University of Hong-Kong, Kowloon.

ABOUT THE AUTHORS

J. Sunday, is a Lecturer with the Department of Mathematics and Statistics, University of Jos, Nigeria. He holds a Ph.D. in Mathematics. His research interests are in Computational Mathematics.

Y. Skwame, is a Lecturer with the Department of Mathematics, Adamawa State University, Mubi-Nigeria. He holds a Ph.D. in Mathematics. His research interests are in Numerical Analysis.

O. Sarjiyus, is a Lecturer with the Department of Computer Science, Adamawa State University, Mubi-Nigeria. He holds an M.Sc. in Computer Science. He is currently a Ph.D. student at the Modibbo Adama University of Technology, Yola-Nigeria. His research interests are in Computer Security.

I. Manga, is a Lecturer with the Department of Computer Science, Adamawa State University, Mubi-Nigeria. He holds an M.Sc. in Computer Science. He is currently a Ph.D. student at the Modibbo Adama University of Technology, Yola-Nigeria. His research interests are in Computational Intelligence.

J.A. Mohammed, is a Lecturer with the Department of Computer Science, Adamawa State University, Mubi-Nigeria. He holds an M.Sc. in Computer Science. His research interests are in Applied Computing and Information Technology.

B.J. Neils, is a Lecturer with the Department of Computer Science, Adamawa State University, Mubi-Nigeria. He holds an M.Sc. in Computer Science. His research interests are in Computer Programming.

SUGGESTED CITATION

Sunday, J., Y. Skwame, O. Sarjiyus, I. Manga, J.A. Mohammed, and B.J. Neils. 2019. "An Efficient Hybrid Algorithm for the Computation of Second-Order Fredholm Integro-Differential Equations". *Pacific Journal of Science and Technology*. 20(1):75-85.

[Pacific Journal of Science and Technology]