A Study of Asymptotic Distribution of Tau (τ) Agreement Index

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ABSTRACT

In agreement measures, we consider the case in which a sample of *n* individuals or subjects is rated independently by two or more raters. There are various measures of agreement amongst which are Cohen's Kappa, Intraclass Kappa, Weighted Kappa, Raw agreement, and Tau (τ) indices just to mention a few. Several authors have argued the case for often-called chance agreement effect, where for example two raters A and B employ different sets of criteria for classifying same objects. In such a case, the observed agreement will be said to be primarily due to chance. This paper has been carried out to observe agreement for the beyond-chance situations using τ statistic amongst other measures of agreement. The asymptotic distribution for estimated τ is derived and its mean and variance obtained. This allows a confidence bound for $\hat{\tau}$ to be proposed. We use some practical examples to determine the confidence bounds across different degrees of freedom.

(Keywords: agreement, confidence bound, beyond chance, asymptotic distribution, Tau statistic, raters, inverse theorem)

INTRODUCTION

Suppose two raters or observers separately classify a sample of subjects using the same measuring scale. The ratings of such observers/raters can be displayed in a square contingency table when studied. Many categorical scales are subjective and this necessitates the assessment of the agreement between the raters involved.

Agreement index has received some attention in categorical data analysis. Authors such as Fleiss (1973); Agresti (1990); Jolayemi (1990); Adebayo and Jolayemi, (1999); Adejumo et al. (2004); and Adejumo, (2005; 2015) have either applied or produced some desirable indices. There are various measures of agreement amongst which are Cohen's Kappa, Intraclass Kappa, Weighted Kappa, Raw agreement, and Tau (τ) indices just to mention a few (Adejumo, 2005).

In this study, the Expected of Tau, E($\hat{\tau}$) and its Variance V($\hat{\tau}$) would be generated from some existing distributions. Tau statistic (τ) as proposed by Jolayemi, (1990) is a statistic for agreement measure that uses the chi-square distribution.

Agreement as described by Williamson and Manatuga (1997) is the extents of correspondence between raters. The degree of such correspondence is always determined by an agreement index or reliability measure (Rueben, 2015). Whenever two raters/observers assign subjects or responses among the same set of categories, measures of agreement for such categorization is possibly developed.

Fleiss (1973) and Jolayemi (1990) claimed that if the degree of agreement between the observer/raters is high, then there is possibility, but not certainly that the classification does reflect actual situation. However, if there is low agreement, then the usefulness of such classification is severely limited.

In view of the primary interests in this paper, we wish to consider how the variance of the estimate of Tau could be derived using the Gamma distribution in relation with the Chi-square distribution. To this effect, this research paper is aimed majorly to show the derivative of both the Mean, E($\hat{\tau}$) and the Variance, V($\hat{\tau}$) of Tau as well as determine the distribution function $\hat{\tau}$ and identify the changes across different degrees of freedom (r).

METHODOLOGY

In this section, we present the derivation of the distribution function, Mean and Variance of Tau statistic obtained from the Gamma distribution with $k = \frac{r}{2}$ and $\lambda = \frac{1}{2}$ which is equivalent to the Chi-square distribution with the use of Laplace transformation.
In addition, we also derive the Confidence bounds for the distribution.

PROOF OF THE DISTRIBUTION OF TAU (τ)

Using the statistic,

$$\hat{\tau} = \pm\sqrt{\frac{\chi^2}{n(c-1)}}, \quad (1)$$

where χ^2 is the Pearson chi-square statistic to independence in the square table.

Recall that if the random variable X has the chi-square distribution, then

$X \sim g(\alpha, \beta)$
$= f(X)$ and

$$f(X) = \frac{\beta^\alpha}{\Gamma(\alpha)} X^{\alpha-1} e^{-\beta X} \qquad X > 0$$

But since for chi-square distribution, $\alpha = \frac{r}{2}, \beta = \frac{1}{2}$ where r being the degrees of freedom then:

$$X \sim g\left(\frac{r}{2}, \frac{1}{2}\right)$$

We assume from (1) that $\chi^2 = g$

Therefore,

$$\hat{\tau} = \pm\sqrt{\frac{g}{n(c-1)}} \quad (2)$$

From Equation (2):

$$g = \hat{\tau}^2 (c-1)n$$

$$\left|\frac{dg}{d\tau}\right| = 2\hat{\tau}(c-1)n$$

$$f(g) = \frac{\left(\frac{1}{2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right)} g^{\frac{r}{2}-1} e^{-\frac{1}{2}g} \qquad g > 0$$

So,

$$f(\hat{\tau}) = f(g)\left|\frac{dg}{d\tau}\right|$$

and g is evaluated in terms of τ.

$$f(\hat{\tau}) = \frac{\left(\frac{1}{2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right)} (\tau^2(c-1)n)^{\frac{r}{2}-1} e^{-\frac{1}{2}\tau^2(c-1)n} . 2\tau(c-1)n$$

$$= \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right)} \left((c-1)n\right)^{\frac{r}{2}} (\hat{\tau}^2)^{\frac{r-2}{2}} . \hat{\tau} e^{-\frac{1}{2}n(c-1)\tau^2}$$

$$f(\hat{\tau}) = \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right)} \left((c-1)n\right)^{\frac{r}{2}} \hat{\tau}^{r-1} e^{-\frac{1}{2}n(c-1)\tau^2} \qquad -1 < \tau < 1 \quad (3)$$

The distribution in Equation (3) is symmetric about zero for any fixed value of *r.*

THE MEAN OF $\hat{\tau}$

Let $A = \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right)} \left((c-1)n\right)^{\frac{r}{2}}$

$$f(\hat{\tau}) = A\hat{\tau}^{r-1} e^{-\frac{1}{2}n(c-1)\tau^2}$$

$$E(\hat{\tau}) = \int_0^\infty \hat{\tau} f(\hat{\tau}) \, d\hat{\tau}$$

$$= \int_0^\infty \hat{\tau} . A\hat{\tau}^{r-1} e^{-\frac{1}{2}n(c-1)\tau^2} \, d\tau$$

$$= A\int_0^\infty \hat{\tau}^r e^{-\frac{1}{2}n(c-1)\tau^2} \, d\tau$$

Let $B = n(c-1)$

$$E(\hat{\tau}) = A\int_0^\infty \hat{\tau}^r e^{-\frac{1}{2}B\tau^2} \quad d\hat{\tau}$$

$$= A\int_0^\infty \hat{\tau}^r e^{-\left(\frac{1}{2}B\hat{\tau}\right)\hat{\tau}} \quad d\hat{\tau}$$

Let $K = \frac{1}{2}B\hat{\tau}$

$D = \left(\frac{1}{2}B\hat{\tau}\right)\hat{\tau}$

E

$(\hat{\tau}) =$

$A\int_0^\infty \tau^r e^{-K\tau} \quad d\tau \qquad (r > -1)$

If, in the integral defining the transform of τ^r,

$$\mathcal{L}\{\tau^r\} = \int_0^\infty e^{-K\tau}\tau^r \quad d\tau$$

$$E(\hat{\tau}) = A\,\mathcal{L}\{\hat{\tau}^r\}$$

We introduce a new variable of integration by setting $D = K\tau$

$$\mathcal{L}\{\tau^r\} = \frac{1}{K^{r+1}}\int_0^\infty D^r e^{-D} \quad dD$$

$$= \frac{1}{K^{r+1}} \;.\Gamma(r+1)$$

$$= \frac{r!}{K^{r+1}}$$

$$E(\hat{\tau}) = A\,.\mathcal{L}[\hat{\tau}^r]$$

$$= \frac{A.r!}{K^{r+1}}$$

$$= \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}\left((c-1)n\right)^{\frac{r}{2}} \;.\; r!}{\Gamma\left(\frac{r}{2}\right).\left(\frac{1}{2}n(c-1)\tau\right)^{r+1}} \qquad \textbf{(4)}$$

THE VARIANCE OF $\hat{\tau}$

$$E(\hat{\tau}^2) = \int_0^\infty \hat{\tau}^2 f(\tau) \quad d\tau$$

$$= A\int_0^\infty \hat{\tau}^2\tau^{r-1} e^{-\frac{1}{2}B\tau^2} \quad d\hat{\tau}$$

$$= A\int_0^\infty \hat{\tau}^{r+1} e^{-\frac{1}{2}B\tau^2} \quad d\hat{\tau}$$

$$= A\int_0^\infty \hat{\tau}^{r+1} e^{-\left(\frac{1}{2}B\tau\right)\tau} \quad d\hat{\tau}$$

$= A\int_0^\infty \hat{\tau}^{r+1} e^{-K\tau} \quad d\hat{\tau} \qquad (r >$
$-1)$

$$= A\,\mathcal{L}\{\hat{\tau}^{r+1}\}$$

$$\mathcal{L}\{\tau^{r+1}\} = \int_0^\infty \hat{\tau}^{r+1}e^{-K\tau} \quad d\hat{\tau}$$

$$= \frac{1}{K^{r+2}}\int_0^\infty D^{r+1} e^{-D} \quad dD$$

$$= \frac{1}{K^{r+2}}\;.\Gamma(r+2)$$

$$= \frac{(r+1)!}{K^{r+2}}$$

$$E(\hat{\tau}^2) = A\,\mathcal{L}\{\tau^{r+1}\}$$

$$= \frac{A\,.(r+1)!}{K^{r+2}}$$

$$V(\hat{\tau}) = E(\hat{\tau}^2) - \left(E(\hat{\tau})\right)^2$$

$$= \frac{A\,.(r+1)!}{K^{r+2}} - \left(\frac{A\,.r!}{K^{r+1}}\right)^2$$

$$= \frac{A\,.(r+1)!}{K^{r+2}} - \frac{A^2\,(r!)^2}{K^{2r+2}}$$

$$= \frac{A\,.K^r\,(r+1)! - A^2(r!)^2}{K^{2r+2}}$$

$$= \frac{2\Gamma\left(\frac{r}{2}\right)\left(\frac{1}{2}\right)^{\frac{r}{2}}\left((c-1)n\right)^{\frac{r}{2}}.\left(\frac{1}{2}n(c-1)\tau\right)^{r}(r+1)! - \left(2\left(\frac{1}{2}\right)^{\frac{r}{2}}\left((c-1)n\right)^{\frac{r}{2}}\right)^2.\,(r!)^2}{\left(\Gamma\left(\frac{r}{2}\right)\right)^2.\left(\frac{1}{2}n(c-1)\tau\right)^{2r+2}} \qquad \textbf{(5)}$$

THE CONFIDENCE BOUNDS FOR $\hat{\tau}$

To construct a confidence bound around $\hat{\tau}$, the expression 100(1- α)% given an approximate 95% confidence interval for $\hat{\tau}$ under Normal approximation:

$$\hat{\tau} \pm Z_{\alpha/2}\sqrt{V(\hat{\tau})}$$

$$\mathbb{P}\left(\hat{\tau} - 1.96\sqrt{V(\hat{\tau})} < \tau < \hat{\tau} + 1.96\sqrt{V(\hat{\tau})}\right) \approx 0.95 \quad \textbf{(6)}$$

We have studied how to use Laplace transformation to establish asymptotic normality of differentiable functions of statistics that are asymptotically normal. Figures 1 and 2 respectively show the plots of r=1 and r=4 for different values of τ. From Figures 1 and 2, we have shown that the asymptotic distribution of a random variable is normal, the two figures for different values of τ are symmetric in shape.

PLOTS FOR VALUES OF τ FOR r=1 and r=4

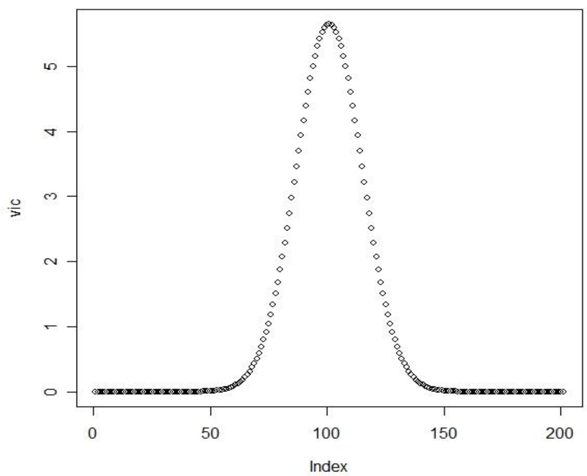

Figure 1: Plot of τ when r=1.

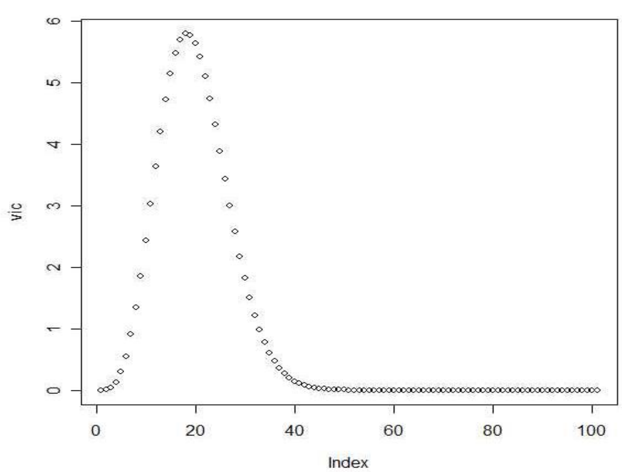

Figure 2: Plot of τ when r=4.

THE MEAN AND VARIANCE OF TAU (τ) WITH $r = (C-1)^2$

$Since\ C = 2\ and\ r = 1$

$$E(\hat{\tau}) = \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}\left((c-1)n\right)^{\frac{r}{2}} . r!}{\Gamma\left(\frac{r}{2}\right).\left(\frac{1}{2}n(c-1)\tau\right)^{r+1}}$$

$$= \frac{2\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\sqrt{\pi}} . n^{\frac{1}{2}} . \frac{1}{\left(\frac{1}{2}n\tau\right)^2}$$

$$= \frac{1.4142\, n^{\frac{1}{2}}}{\sqrt{\pi}\left(\frac{1}{2}n\tau\right)^2}$$

$$E(\hat{\tau}^2) = \frac{2\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} . B^{\frac{1}{2}} . \frac{2}{\left(\frac{1}{2}B\tau\right)^3}$$

$Since\ c = 2,\ B = n(c-1) = n(2-1) = n$

$$= \frac{2\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\sqrt{\pi}} . n^{\frac{1}{2}} . \frac{2}{\left(\frac{1}{2}n\tau\right)^3}$$

$$= \frac{2.8284\, n^{\frac{1}{2}}}{\sqrt{\pi} . \left(\frac{1}{2}n\tau\right)^3}$$

$$V(\hat{\tau}) = E(\hat{\tau}^2) - \left(E(\hat{\tau})\right)^2$$

$$= \frac{2.8284\, n^{\frac{1}{2}}}{\sqrt{\pi} . \left(\frac{1}{2}n\tau\right)^3} - \left(\frac{1.4142\, n^{\frac{1}{2}}}{\sqrt{\pi}\left(\frac{1}{2}n\tau\right)^2}\right)^2$$

$$= \frac{2.8284\, n^{\frac{1}{2}}}{\sqrt{\pi} . \left(\frac{1}{2}n\tau\right)^3} - \frac{2\, n}{\pi\left(\frac{1}{2}n\tau\right)^4}$$

$$= \frac{\sqrt{\pi} . 1.4142\, n^{\frac{1}{2}} . n\tau - 2n}{\pi\left(\frac{1}{2}n\tau\right)^4}$$

$$V(\hat{\tau}) = \frac{1.4142\,(\pi n)^{\frac{1}{2}} . n\tau - 2n}{\pi\left(\frac{1}{2}n\tau\right)^4}$$

$Since\ c = 3\ and\ \ r = 4$

$$A \quad = \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right)}\left((c-1)n\right)^{\frac{r}{2}} = \frac{2\left(\frac{1}{2}\right)^{2}}{\Gamma(2)}\left((3-1)n\right)^{2} = 2n^{2}$$

$$B \quad = n(c-1) = n(3-1) = 2n$$

$$K \quad = \frac{1}{2}B\tau = \frac{1}{2}(2n)\tau = n\tau$$

$$\mathrm{E}(\hat{\tau}) = \frac{A.r!}{K^{r+1}} = \frac{2n^{2}}{(n\tau)^{4+1}}.4! = \frac{48n^{2}}{n^{5}\tau^{5}} = \frac{48}{n^{3}\tau^{5}} \quad \textbf{(7)}$$

$$\mathrm{E}(\tau^{2}) = \frac{A(r+1)!}{K^{r+2}} = \frac{2n^{2}}{(n\tau)^{4+2}}.5! = \frac{240n^{2}}{n^{6}\tau^{6}} = \frac{240}{n^{4}\tau^{6}}$$

$$\mathrm{V}(\hat{\tau}) = \frac{240}{n^{4}\tau^{6}} - \left(\frac{48}{n^{3}\tau^{5}}\right)^{2}$$

$$= \frac{240}{n^{4}\tau^{6}} - \frac{2304}{n^{6}\tau^{10}}$$

$$= \frac{240n^{2}\tau^{4} - 2304}{n^{6}\tau^{10}} \quad \textbf{(8)}$$

Let there be two random variables *x* and *y* with probability density distributions $f(x)$ *and* $h(y)$ and if $x = y$ then by inverse theorem $f(x) = h(y)$ (Conrant, 1988; Francis and Hildebrand 1990; Scott and Tims, 1994; *Hazewinkel, 2001).*

Implication: Comparing the distribution of estimated τ in Equation (3) with the Normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma^{2}}\, e^{-\frac{1}{2\sigma^{2}}(x-\mu)^{2}} \qquad -\infty < x < \infty$$

If $\mu = 0$, then $\sigma^{2} = \frac{1}{(c-1)n}$

Applications and Discussion

Illustration 1: The data in respect of registered TB patients at the National Tuberculosis and Leprosy Training Center, Zaria are given below to study the agreement between Sputum Test and X-Ray. (Source: National Tuberculosis and Leprosy Programme, 4th Edition Manual (NTBLCP, 2006))

Sputum Test

		+	-	
X-Ray	+	224	179	403
	-	0	96	96
		224	275	499

$$H_{0}: \pi_{ij} = \pi_{i.}\pi_{.j} \qquad \forall \ \ i \neq j$$

$$H_{1}: \pi_{ij} \neq \pi_{i.}\pi_{.j}$$

$$X^{2} = \sum \frac{n_{ij}^{2}}{e_{ij}} - N$$

$$= \frac{224^{2}}{181} + \frac{179^{2}}{222} + \ldots + \frac{96^{2}}{53} - 499$$

$$= 96.44$$

So,

$$\tau = \pm\sqrt{\frac{x^{2}}{n(c-1)}}$$

$$= \pm\sqrt{\frac{96.44}{499\,(2-1)}}$$

$$= \pm 0.4396 \approx \pm 0.44$$

Therefore, the strength of agreement between Sputum Test and X-Ray with $\hat{\tau} = 0.44$ is moderate.

$$\mathrm{E}(\hat{\tau}) = \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}\left((c-1)n\right)^{\frac{r}{2}}.\, r!}{\Gamma\left(\frac{r}{2}\right).\left(\frac{1}{2}n(c-1)\tau\right)^{r+1}}$$

$$Since \ \ r = 1 \ and \ c = 2,$$

$$= \frac{1.4142\, n^{\frac{1}{2}}}{\sqrt{\pi}\left(\frac{1}{2}n\tau\right)^{2}}$$

$$E(\hat{\tau} = 0.44) = \frac{1.4142 \times (\sqrt{499})}{\sqrt{3.412} \times \left(\frac{1}{2} \times 499 \times .044\right)^2}$$

$$= 0.0015$$

By Inverse Theorem,

$$E(\hat{\tau}) = 0$$

$$V(\hat{\tau}) = \frac{1.4142\,(\pi n)^{\frac{1}{2}}.n\tau - 2n}{\pi\left(\frac{1}{2}n\tau\right)^4}$$

$$V(\hat{\tau} = 0.44) = \frac{1.4142\,(\pi n)^{\frac{1}{2}}.n\tau - 2n}{\pi\left(\frac{1}{2}n\tau\right)^4}$$

$$= \frac{\left(1.4142 \times \sqrt{(3.412 \times 499)} \times (499 \times 0.44)\right) - (2 \times 499)}{3.412 \times \left(\frac{1}{2} \times 499 \times 0.44\right)^4}$$

$$= 0.0000248$$

By Inverse Theorem,

$$V(\hat{\tau}) = \frac{1}{(c-1)n} = \frac{1}{(2-1)499} = \frac{1}{499} = 0.002004$$

$$S.D.(\hat{\tau} = 0.44) = \sqrt{0.0000248} = 0.00498$$

To construct a confidence bound for $\hat{\tau} = 0.44$

$$\hat{\tau} \pm Z_{\alpha/2}\sqrt{Var(\hat{\tau})}$$

$$= 0.44 \pm 1.96 \times 0.00498$$

$$= 0.44 \pm 0.00976$$

$$\therefore \hat{\tau} = (0.43\,, 0.45)$$

Illustration 2: The data for this exercise relate the relationship between people's attitude on the government's role in guaranteeing jobs and their votes in state and local elections (Lawal, 2003).

		Attitude			
		Liberal	Moderate	Conservative	
Vote	Democratic	312	34	115	461
	Even	159	24	110	293
	Republican	210	32	137	379
		681	90	362	1133

$$H_0: \pi_{ij} = \pi_{i.}\pi_{.j} \qquad \forall \;\; i \neq j$$

$$H_1: \pi_{ij} \neq \pi_{i.}\pi_{.j}$$

$$X^2 = \sum \frac{n_{ij}^2}{e_{ij}} - N$$

$$= \frac{312^2}{277} + \frac{34^2}{37} + \ldots + \frac{137^2}{121} - 1133$$

$$= 19.71$$

Since,

$$\tau = \pm\sqrt{\frac{x^2}{n(c-1)}}$$

$$= \pm\sqrt{\frac{19.71}{1133\,(3-1)}}$$

$$= \pm 0.0933 \approx \pm 0.09$$

Therefore, the strength of agreement between people's Attitude and their Votes with $\hat{\tau} = 0.09$ is poor.

$$E(\hat{\tau}) = \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}\left((c-1)n\right)^{\frac{r}{2}}.\,r!}{\Gamma\left(\frac{r}{2}\right).\left(\frac{1}{2}n(c-1)\tau\right)^{r+1}}$$

$$Since\; r = 4\; and\; c = 3,$$

$$E(\hat{\tau}) = \frac{48}{n^3\tau^5}$$

$$E(\hat{\tau} = 0.09) = \frac{48}{((1133)^3 \times (0.09)^5)}$$

$$= 0.0056$$

By Inverse Theorem,

$$E(\hat{\tau}) = 0$$

$$V(\hat{\tau}) = \frac{240n^2\tau^4 - 2304}{n^6\tau^{10}}$$

$$V(\hat{\tau} = 0.09) = \frac{(240 \times (1133)^2 \times (0.09)^4) - 2304}{((1133)^6 \times (0.09)^{10})}$$

$$= 0.000243$$

By Inverse Theorem,

$$V(\hat{\tau}) = \frac{1}{(c-1)n} = \frac{1}{(3-1)1133} = \frac{1}{2266} = 0.00044$$

$$S.D(\hat{\tau} = 0.09) = \sqrt{0.000243} = 0.01558$$

To construct a confidence bound for $\hat{\tau} = 0.09$

$$\hat{\tau} \pm Z_{\alpha/2}\sqrt{Var(\hat{\tau})}$$

$$= 0.09 \pm 1.96 \times 0.01558$$

$$= 0.09 \pm 0.0305$$

$$\therefore \hat{\tau} = (0.06\,, 0.12)$$

CONCLUSION

At the end of the derivations in the previous section, the distribution of Tau is:

$$f(\tau) = \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}\right)}\left((c-1)n\right)^{\frac{r}{2}}\tau^{r-1}e^{-\frac{1}{2}n(c-1)\tau^2}$$

where,

r is the degree of freedom for any of the squared contingency table under consideration whereby r could only be a positive integer i.e. $r > 0$.

c is the number of columns for the squared contingency table been considered, this also could only be a positive integer where $c \geq 2$.

n is the total number of observations considered, that is, the sum of either row total or column total of the squared contingency table.

$\hat{\tau}$ is the estimate of Tau from the statistic $\pm\sqrt{\frac{x^2}{n(c-1)}}$, for the distribution of Tau, we only consider the absolute of Tau i.e. $|\tau|$

where $-1 < \tau < 1$ (see Table 1 in the Appendix).

Based on the derivation, we concluded that the Mean of Tau is:

$$E(\hat{\tau}) = \frac{2\left(\frac{1}{2}\right)^{\frac{r}{2}}\left((c-1)n\right)^{\frac{r}{2}} \cdot r!}{\Gamma\left(\frac{r}{2}\right) \cdot \left(\frac{1}{2}n(c-1)\tau\right)^{r+1}}$$

Also, we made a conclusion on the Variance of Tau to be:

$$V(\hat{\tau}) =$$

$$\frac{2\Gamma\left(\frac{r}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{\frac{r}{2}}\left((c-1)n\right)^{\frac{r}{2}} \cdot \left(\frac{1}{2}n(c-1)\tau\right)^{r}(r+1)! - \left(2\left(\frac{1}{2}\right)^{\frac{r}{2}}\left((c-1)n\right)^{\frac{r}{2}}\right)^2 \cdot (r!)^2}{\left(\Gamma\left(\frac{r}{2}\right)\right)^2 \cdot \left(\frac{1}{2}n(c-1)\tau\right)^{2r+2}}$$

The compact form of Variance is:

$$V(\hat{\tau}) = \frac{2\Gamma\left(\frac{r}{2}\right)\left(\frac{n}{2}\sqrt{r}\right)^{\frac{r}{2}}\tau^r.(r+1)! - (2r!)^2}{\left(\Gamma\left(\frac{r}{2}\right)\right)^2 \cdot \left(\frac{n}{2}\sqrt{r}\right)^{r+2}\tau^{2r+2}}$$

where,

$$\Gamma\left(\frac{r}{2}\right) = \left(\frac{r-2}{2}\right)!$$

if r is Even

or

$$\Gamma\left(\frac{r}{2}\right) = \frac{(r-2)(r-4)\ldots 1}{2^{(r-1)/2}}\sqrt{\pi}$$

if r is Odd

Comparing the distribution of estimated τ with the Normal distribution and $\mu = 0$ (an assumption), the approximated Variance of τ, $\sigma^2 = \frac{1}{(c-1)n}$.

At the end of the derivations, we observed the confidence bounds of Tau to be $\hat{\tau} \pm Z_{\alpha/2}\sqrt{V(\hat{\tau})}$ Therefore,

$$\mathbb{P}\left(\hat{\tau} - 1.96\sqrt{V(\hat{\tau})} < \tau < \hat{\tau} + 1.96\sqrt{V(\hat{\tau})}\right) \approx 0.95$$

This is an approximate 95% confidence interval for τ under Normal approximation.

Also, the confidence bound of $\hat{\tau}$ in the first and second applications are (0.43, 0.45) and (0.06, 0.12), respectively. This means that we can assert with a probability of 0.95 that the random end-points will enclose the true unknown values of τ.

CONFLICTS OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

Appendix

Table 1: Range of $|\tau|$ Statistic with the Respective Strength of Agreement.

$\lvert\tau\rvert$ Statistic	Strength Of Agreement
0.00– 0.20	Poor
0.21 – 0.40	Slight
0.41 – 0.60	Moderate
0.61 – 0.80	Substantial
0.81 – 1.00	Almost perfect

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