

Variation of Exit Gradient Downstream of Weirs on Permeable Foundations.

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ABSTRACT

Design of weirs on permeable foundations requires the estimation of exit gradients after the end of downstream pile on the extended floor length, provided for safety purposes. Khosla et al. (1981) had suggested curves for the pile at the downstream end of the impervious floor of the weir. However, if some additional floor length is provided after the end sill in the form of filter or downstream protection such as stones underlain by T-V layer, then in this situation, no ready to use curves are easily available for calculating the exit gradient. The present work is aimed as an attempt in this direction to develop new curves so that the safety of the weir can be decided on the basis of exit gradient.

(Keywords: weir, exit gradient, foundations, construction engineering, safety, curve calculation)

THEORETICAL SOLUTION BY SCHWARZ-CHRISTOFFEL TRANSFORMATION

Study on Weirs Having Aprons with Intermediate Pile

Figure 1 (a) shows schematically a case of an impervious apron on a permeable foundation with an intermediate pile. The solution of pressure distribution along the apron is obtained by Schwarz-Christoffel transformation by Khosla (1981) and Leliavsky (1959) of the flow profile as done by Khosla et al. in the complex z zone in Figure 1(a) to the complex ζ zone as in Figure 1(b).

The general equation of Schwarz-Christoffel transformation is:

$$z = A \int \frac{d\zeta}{(\zeta - \zeta_1)^{\lambda_1} (\zeta - \zeta_2)^{\lambda_2} (\zeta - \zeta_3)^{\lambda_3} \dots} \quad (1)$$

In which, $\pi \lambda_1, \pi \lambda_2, \pi \lambda_3, \dots$ etc., are the changes in angles of the polygon which are required to transform it into a straight line and $\zeta_1, \zeta_2, \zeta_3, \dots$ etc., are the coordinates of its vertices. A is an arbitrary constant.

Referring to Figure 1 (a), for the polygon BCDEA, it can be seen that:

$$\lambda_1 = \frac{\pi/2}{\pi} = \frac{1}{2};$$

$$\lambda_2 = \frac{-\pi}{\pi} = -1,$$

$$\text{and } \lambda_3 = \frac{\pi/2}{\pi} = \frac{1}{2}.$$

The coordinates of the vertices in Figure 1(b), ζ_1, ζ_2 and ζ_3 are 1, K , and -1 , respectively corresponding to points C', D', and E'.

Hence the transformation will be reduced to:

$$\begin{aligned} z &= A \int \frac{d\zeta}{(\zeta - 1)^{1/2} (\zeta - K)^{-1} (\zeta + 1)^{1/2}} \\ &= A \int \frac{(\zeta - K) d\zeta}{\sqrt{(\zeta^2 - 1)}} \end{aligned} \quad (2)$$

Integrating Equation (2):

$$z = A \sqrt{\zeta^2 - 1} - A K \ln (\zeta - \sqrt{\zeta^2 - 1}) + B \quad (3)$$

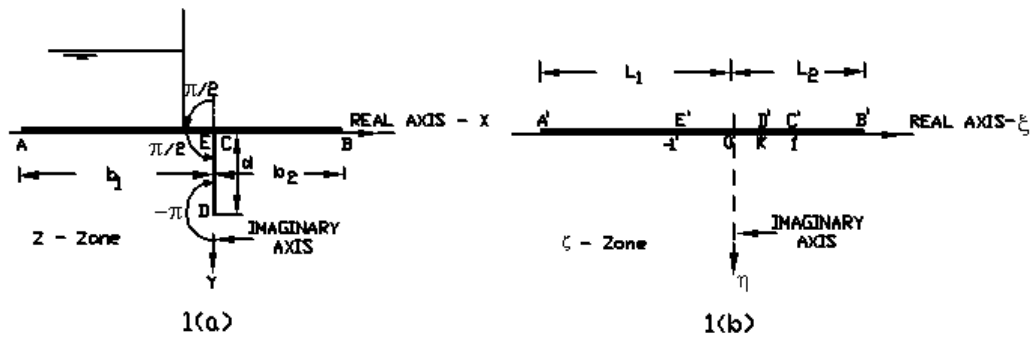


Figure 1: Weir with Intermediate Pile.

Referring to Figure 1(a), at C, $z = x + iy = 0$ and at C', $\zeta = 1$.

Substituting in Equation (3),

$$B = 0 \quad (4)$$

At E in Figure 1(a), $z = 0$ and at E' in Figure 1(b), $\zeta = -1$. Substituting $B = 0$ in Equation (3), $K = 0$.

Therefore Equation (3) reduces to:

$$z = A \sqrt{\zeta^2 - 1} \quad (5)$$

At D in Figure 1(a), $z = id$ and at D' in Figure 1(b) $\zeta = K = 0$. Hence from Equation (5) $a = d$ and,

$$z = d \sqrt{\zeta^2 - 1} \quad (6)$$

The points A and A' have the coordinates $z = -b_1$ and $\zeta = -L_1$, respectively.

Hence, $b_1 = d \sqrt{L_1^2 - 1}$ or,

$$L_1 = \sqrt{1 + (b_1/d)^2} = \sqrt{1 + \alpha_1^2} \quad (7)$$

where $\alpha_1 = b_1/d$

Points B and B' have the coordinates $z = b_2$ and $\zeta = L_2$

Hence $b_2 = d \sqrt{L_2^2 - 1}$, i.e.,

$$L_2 = \sqrt{1 + (b_2/d)^2} = \sqrt{1 + \alpha_2^2} \quad (8)$$

$$\begin{aligned} \text{Total length of the transformed apron} \\ = L_1 + L_2 \end{aligned} \quad (9)$$

Study of Weir Having Apron with Pile at End

If in Figure 1 (a) when a change is made by making $b_2 = 0$, the case of apron having pile at the end as in Figure 2 (a) in z zone can result. Noting that $K = 0$, the transformed figure in ζ zone is shown in Figure 2(b). The transformation equation will be the same as (6) i.e.,

$$z = d \sqrt{\zeta^2 - 1} \quad (6)$$

and

$$L_1 = \sqrt{1 + (b_1/d)^2} = \sqrt{1 + \alpha_1^2} = \sqrt{1 + \alpha^2} \quad (10)$$

since $b_2 = 0$,

$$L_2 = \sqrt{1 + (b_2/d)^2} = \sqrt{1 + \alpha_2^2} = 1. \quad (11)$$

Thus the total length of apron will be $L_1 + 1$

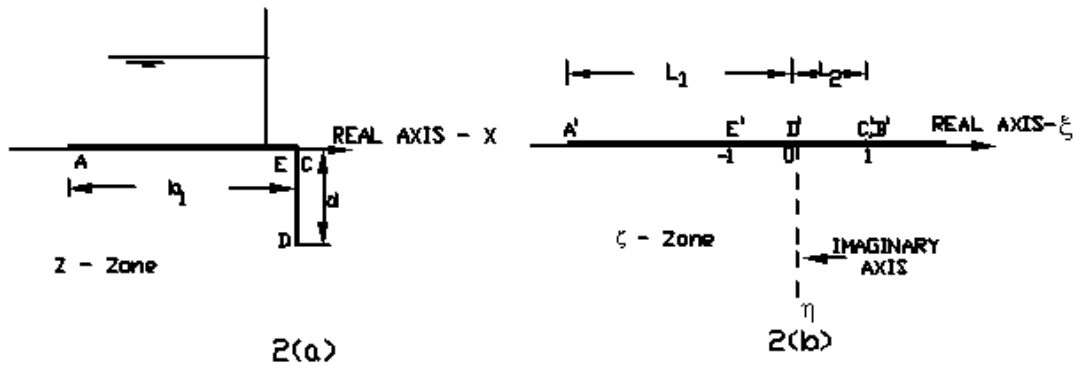


Figure 2: Weir with Pile at the End.

Stream Lines and Equipotentials with Pile at End of Apron

If b is taken as the length of the apron in the ζ zone, a set of confocal hyperbolae and ellipses can be formed which can represent the equipotentials and streamlines with a floor without any pile. The equation of the transformation required to get the set of curves is $z = \frac{b}{2} \cosh w$ where w is the complex potential, $w = u + i v$.

If u is a constant it can represent a stream line and if v is a constant, it can represent a equipotential. Referring to Figure 1 (b), if b is taken as the width of apron a correction of coordinate is required. $z = \zeta + \frac{L_1 - 1}{2}$.

If we let $\frac{L_1 + 1}{2} = \lambda$ and $\frac{L_1 - 1}{2} = \lambda_1$ the transformation required will be:

$$\zeta = \lambda \cosh w - \lambda_1 \tag{12}$$

Since the problem at hand is an apron with a pile at the end, the transformation required can be obtained from Equation (6) $z = d \sqrt{\zeta^2 - 1}$, which will reduce to:

$$z = d \sqrt{(\lambda \cosh w - \lambda_1)^2 - 1} \tag{13}$$

where $\lambda = \frac{L_1 + 1}{2}$ and $\lambda_1 = \frac{L_1 - 1}{2}$.

Khosla et al. has worked out the equation for potential distribution on the floor before the apron as $P_E = \frac{H}{\pi} \cos^{-1} \frac{\lambda - 2}{\lambda}$ and the exit gradient after the apron end as in Figure 2(a) as:

$$G_E = \frac{H}{\pi d} \frac{1}{\sqrt{\lambda}} \tag{14}$$

It is required to find the exit gradient downstream of the impervious apron end. From Equation (7),

$$\frac{z^2}{d^2} = (\lambda \cosh w - \lambda_1)^2 - 1 \tag{15}$$

where $z = x + i y$ and $w = u + i v$

Expanding Equation (15) and equating the imaginary parts on both sides of the equation and rearranging,

$$\lambda^2 \sin 2v \cdot \cosh u \cdot \sinh u - 2\lambda \lambda_1 \sinh u \cdot \sin v = \frac{2 x y}{d^2}$$

For a specified value u_1 for u , the equation can be written as:

$$\frac{\lambda^2}{2} \sin 2v \cdot \sinh 2u_1 - 2 \lambda \lambda_1 \sinh u_1 \cdot \sin v = \frac{2 x y}{d^2}$$

Differentiating the above equation w.r.t. y ,

$$(\lambda^2 \sinh 2u_1 \cdot \cos 2v - 2 \lambda \lambda_1 \sinh u_1 \cdot \cos v) \frac{dv}{dy} = \frac{2}{d^2} (x + y \frac{dx}{dy})$$

For a potential function $v = 0$ that is at the exit, $y = 0$ and hence,

$$\left(\frac{dv}{dy} \right)_{y=0} = \frac{2 x}{d^2} \cdot \frac{1}{(\lambda^2 \sinh 2u_1 - 2 \lambda \lambda_1 \sinh u_1)}$$

The potential function $v = \pi \times \frac{P}{H}$ where p is the potential head and H is the head of the weir.

Hence,

$$\frac{dv}{dy} = \frac{\pi}{H} \left(\frac{dp}{dy} \right)$$

$$G_E = \frac{dp}{dy} = \frac{H}{\pi} \frac{dv}{dy}, \text{ or}$$

$$G_E = \frac{2 H}{\pi d} \frac{x}{d} \frac{1}{(\lambda^2 \sinh 2u_1 - 2 \lambda \lambda_1 \sinh u_1)} \quad (16)$$

At the downstream bed, $v = 0$ and $y = 0$ and hence from Equation (15):

$$\frac{x^2}{d^2} = (\lambda \cosh u_1 - \lambda_1)^2 - 1 \quad (17)$$

In an illustrative problem one can select various values of $\alpha = \frac{b}{d}$ and investigate the variation of

the exit gradient along the distance $\frac{x}{d}$ from the apron end.

In Equations (16) and (17), $\lambda = \frac{L_1 + L_2}{2} = \frac{1}{2} (1 + \sqrt{1 + \alpha^2})$ and $\lambda_1 = \lambda - 1$.

For any value of α assumed, a value of u_1 can be assumed and $\frac{x}{d}$ can be calculated using Equation (17).

Making use of that value of $\frac{x}{d}$, by using Equation

(16) the parameter $\frac{G_E}{H/d}$ can be calculated. The

exit gradient at the end of apron can be calculated using Khosla's result $G_E = \frac{H}{\pi d} \times \frac{1}{\sqrt{\lambda}}$

or obtained by using the formula (17) and (16) for very low value of u_1 like 0.001. The result is depicted in Figure 3 in a non-dimensional form.

Critical Exit Gradients when Protected by Stones Enclosed in Nets

A launching apron provided downstream of the weirs reduces the risk of failure of the weir when the flood is unusually high as it has the capacity to adjust to any scour profile if it is formed. A bed of stones enclosed in nets are more resistant to scour because the stones act in unison and are held together by the nets as suggested by Posey (1961), Pillai (1977), and Pillai et al. (2008).

A filter at the bottom of the stones enclosed in nets can be provided as per the Terzaghi Vicksberg specifications USWES (1945) or the filter can be a geo-fabric. In both the cases, the sand underneath will be effectively prevented from escaping through the interstices of the stones enclosed in nets. The net at the apron end can be effectively anchored to the concrete apron so that it does not get displaced.

The seepage force in the sand underneath will be given by $i \gamma_w AT$ where i is the upward gradient, γ_w is the specific weight of water A is the area sand column and T is the thickness on which the upward gradient acts.

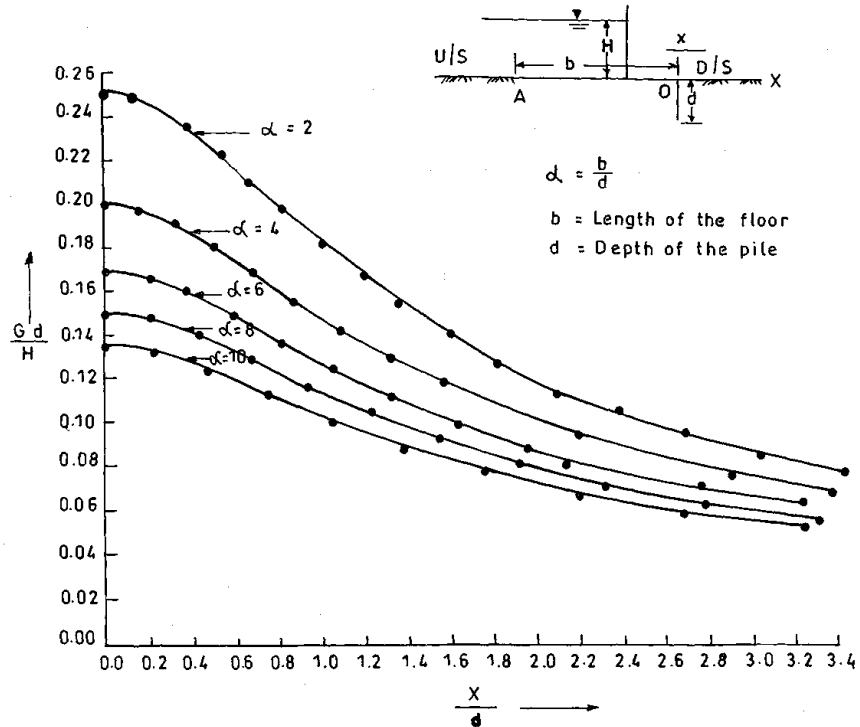


Figure 3: Variation of Exit Gradient Downstream of End Pile.

Since the filter and the stones enclosed in nets are highly permeable the upward gradient on it can be neglected in comparison.

Considering the stability of a sand column after the end of apron, the resisting forces coming into play are the submerged weight of the soil and the stones enclosed in nets

$\frac{G-1}{1+e} \gamma_w AT$ where T is the total thickness of sand protection given.

The specific gravity of the sand particles and stones is nearly equal to 2.6 and e is the void ratio nearly equal to 0.6. In addition, a stabilizing force is the friction of the soil on the piles.

The seepage force in the sand underneath will be given by $i \gamma_w A (T-t)$ where A is the area of base of the sand column, T is the total, thickness of sand column and protection given, t is the thickness of protection, i is the upward gradient, γ_w is the specific weight of water. Since the filter and the stones enclosed in nets are highly

permeable the upward gradient on it can be neglected in comparison.

Hence the critical exit gradient will be more than $\frac{(G-1) \gamma_w AT}{(1+e) i \gamma_w A (T-t)}$, i.e. greater than $\frac{T}{T-t}$.

The result is encouraging since a protection with stones enclosed in nets underlain by a filter becomes very effective against failure and the exit gradient can be greater than 1 (Rao, 1991). It is also significant that the exit gradient gets reduced significantly at a small distance from the apron end of $\frac{x}{d} = 2$ as shown in Figure 3.

EXPERIMENTS TO SHOW SAFETY AGAINST HIGHER EXIT GRADIENTS

Some model experiments were conducted to illustrate the safety against high exit gradients when the sand bed is protected by stones enclosed in nets. The experiments were done at

the Hydraulics Laboratory of NIT, Kurukshetra, India (Rao, 1991). The model was made in masonry flume 23 cm wide 75 cm deep and 2.5 m long with glass on one side. The bed 30 cm deep was made of sand passing through 1.8 mm size sieve ($D_{15} = 0.21$ mm; $D_{50} = 0.355$ mm; and $D_{85} = 0.6$ mm). An apron was made using a wooden plank 2.5 cm thick and of required length. A vertical plate was fitted on the apron to retain water to a maximum depth of 43 cm. A cutoff of desired depth could be fitted at the end of apron. The arrangement has been shown schematically in Figure 4.

On the downstream of apron a bed of stones enclosed in thin polypropylene nets of thickness 1.5 cm (stone size 4.75 mm) underlain by a sand filter 1 cm thick is provided to any length when required. The filter is of sand ($D_{15} = 1.2$ mm; $D_{50} = 1.85$ mm). The filter material satisfied the Terzaghi Vicksberg specifications⁷ as verified below:

$$(a) \frac{D_{15} \text{ of filter}}{D_{85} \text{ of base}} = \frac{1.2 \text{ mm}}{0.6 \text{ mm}} = 2.0 < 5$$

$$(b) \frac{D_{15} \text{ of filter}}{D_{15} \text{ of base}} = \frac{1.2 \text{ mm}}{0.21 \text{ mm}} = 5.71$$

(between 4 and 20)

$$(c) \frac{D_{50} \text{ of filter}}{D_{50} \text{ of base}} = \frac{1.85 \text{ mm}}{0.355 \text{ mm}} = 5.2 < 25.$$

An inlet pipe with a control valve admitted water to the flume on the upstream side and the water is discharged from the flume by a pipe with control valve on the downstream. The discharges could be adjusted to maintain any desired head upstream of the weir and to maintain the water at bed level on the downstream.

The weir profile was made carefully and tested whether it is safe or not. If it is safe for an arrangement for some time, the head on the upstream is progressively increased to test its safety. When the model failed, the model is dismantled, sand is taken out and a new arrangement is remade for testing. The following table gives results from some tests.

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$$\alpha = b/d; \quad \lambda = \frac{1}{2} (1 + \sqrt{1 + \alpha^2});$$

G_1 – exit gradient at the end of apron. G_2 –exit gradient at end of the protection.

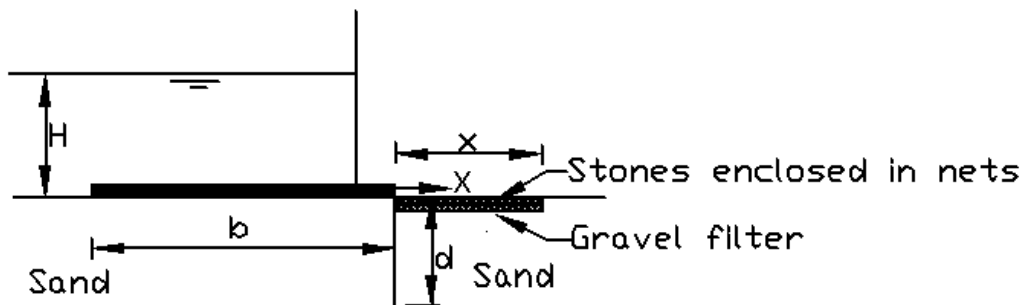


Figure 4: Schematic Diagram Showing the Arrangement of Weir Model.

Table 1: Experimental Results of Tests on Weir.

Expt No:	Apron Length b cm	Pile Depth d cm	Head of Water H cm	$\alpha = b/d$	λ	$\frac{G_1 d}{H}$	G_1	X cm	X/d	$\frac{G_2 d}{H}$	G_2	Remarks
1	34	12.5	42	2.72	1.95	0.228	0.77	0	0	-	-	safe
2	25	12.5	42	2.0	1.62	0.25	0.84	10	0.8	0.2	0.67	safe
3	25	6	30	4.17	2.64	0.2	1.0	0	0	-	-	failed
4	25	6	40	4.17	2.64	0.2	1.33	10	1.67	0.12	0.8	safe
5	25	2.5	18	10	5.52	0.135	0.97	0	0	-	-	failed
6	10	2.5	16	4	2.56	0.2	1.28	3.5	1.4	0.13	0.83	safe

In Experiment 1, no additional protection was used after the end of apron and it worked safely. The exit gradient at the end of apron was 0.77 much below the critical value of 1.

In Experiment 2, a protection by stones enclosed in nets was placed, the exit gradient at the end of apron was higher (0.84) but below 1 and the exit gradient at the end of protection is reduced to 0.67. The arrangement worked safely.

In Experiment 3, the pile depth is reduced and the weir failed at a lower head of 30 cm when the exit gradient at the end of apron is about 1.

In Experiment 4, a protection of stones enclosed in nets to a length of 10 cm was provided at the end of apron. The weir was safe. It may be noted that the exit gradient at the end of apron is as high as 1.33 and still the weir was safe. This was due to the protection by stones enclosed in net and also friction afforded by soil against failure at

the end. The exit gradient at $\frac{x}{d} = 0.95$ can be seen to be equal to 1.

In Experiment 5, no protection is provided after the apron and weir failed when the exit gradient at the end of apron was about 1.

In Experiment 6, a protection was provided and b/d ratio was only 4. The exit gradient at the end of apron was 1.28 and still the weir was safe.

The exit gradient at $\frac{x}{d} = 0.872$ can be seen to be equal to 1.

DISCUSSION

The case of a weir with an intermediate pile was considered first and transformation equation was derived. The case of a weir with pile at the end was considered as a special case when the length $b_2 = 0$. Thus it was ensured that the proper angles for polygon BCDEA were taken. By considering the floor length as $L_1 + 1$, the bifocal set of ellipses and hyperbolas as streamlines and equipotentials are envisaged.

Equations (11) and (10) can be used to calculate the exit gradient. For any particular value of α , by assuming a very small value of $u_1 = 0.001$ the exit gradient is calculated and it was found to be the same as given by Khosla¹ for exit gradient at

the apron end i.e., $G_E = \frac{H}{\pi d} \frac{1}{\sqrt{\lambda}}$ (8)

validating the use of equations (17) and (16).

The exit gradient at distance a parameter $\frac{x}{d}$ from the end of apron is worked out by considering the vertical gradient at various points on the equipotential line, $v = 0$. The relationship could be expressed as a function of non-dimensional parameters. The results show that the exit gradient falls rapidly to about half the value at a distance of $\frac{x}{d} = 2$.

Experiments show that the failure of weir without protection after the apron occurs when the exit gradient is nearly equal to 1. The weir models were safe for higher exit gradients at apron end when stones enclosed in nets are provided.

The safety can partly be due to the friction of sand on the sides near the pile. The stones enclosed in nets can be continued to a distance where the exit gradient is within safe limits.

The analysis does not imply that the design of the weir should be in the critical condition of limiting exit gradient. The design should incorporate sufficient factors of safety as per design practice. But advantage can be taken for a greater exit gradient possible at the apron end when stones enclosed in nets are given as additional protection.

CONCLUSIONS

For any value of the stream function u_1 , the distance from the apron could be calculated using the Equation (11). When $u \cong 0$, it gives the exit gradient at the end of apron having pile at the end. The value is found to be same as given by

equation $G_E = \frac{H}{\pi d \sqrt{\lambda}}$...given by Khosla.

When, $u = u_1$ has a higher value, the corresponding value of $\frac{x}{d}$ from Equation (10)

and later value of $\frac{G_E d}{H}$ can be calculated using Equation (16).

The non-dimensional curve Figure 3 shows that exit gradient decreases considerably with $\frac{x}{d}$ and

when $\frac{x}{d} > 2$, the exit gradient is nearly reduced to half the value.

Protection using stones enclosed in nets are effective in preventing surface erosion and it also offers safety against high exit gradients. The critical exit gradients with such protection can be higher than 1 since seepage force on the stones and filter will be negligible and also because of friction of neighboring sand.

Since the exit gradient reduces rapidly with $\frac{x}{d}$, the protection need be continued only for short distance from the apron end where the exit gradient is within safe limits.

The results suggest only that while the design of weirs incorporates proper factor of safety, the additional safety of providing a bed of stones enclosed in nets after the apron end may be considered for some distance.

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