On Equivalence of n-Norms in n-Normed Spaces.

B. Surender Reddy[1] and Hemen Dutta[2]

[1]Department of Mathematics, PGCS, Saifabad, Osmania University, Hyderabad-500004, AP, India.
[2]Department of Mathematics, Gauhati University, Kokrajhar Campus, Assam, India.

E-mail: bsrmathou@yahoo.com[1]
hemen_dutta08@rediffmail.com[2]

ABSTRACT

In this paper, we investigate some properties of linear n-normed spaces and obtained necessary and sufficient conditions for n-norms to be equivalent on linear n-normed spaces.

(Keywords: n-normed space, Cauchy sequence, equivalent norms)

(AMS Subject Classification No: 46B20, 46C05)

INTRODUCTION

In [6, 7] Gähler introduced an attractive theory of 2-norm and n-norm on a linear space. Raymond W. Freese and Y.J. Cho [4] introduced as a survey of the latest results on the relations between linear 2-normed spaces and normed linear spaces, completion of linear 2-normed spaces. A systematic development of linear n-normed spaces has been extensively made by S.S. Kim and Y.J. Cho [8], R. Malceski [5], A. Misiak [1] and Hendra Gunawan and Mashadi [3]. In this paper, some necessary and sufficient conditions for n-norms to be equivalent on a linear n-normed space are given.

Let $n \in N$ and let X be a real linear space of dimension $d \geq n$. A real valued function $\|\bullet,\bullet,\ldots,\bullet\| : X^n \to R$ satisfying the following four properties:

nN_1: $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent vectors

nN_2: $\|x_1, x_2, \ldots, x_n\| = \|x_{j_1}, x_{j_2}, \ldots, x_{j_n}\|$ for every permutation $(j_1, j_2, \ldots, j_n)$ of $(1,2,\ldots,n)$

i.e., $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.

nN_3 : $\|x_1, x_2, \ldots, x_{n-1}, \alpha x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in R$

nN_4: $\|x_1, x_2, \ldots, x_{n-1}, y+z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$ for all $y, z, x_2, \ldots, x_{n-1} \in X$, is called an n-norm on X and the pair $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ is called a linear n-normed space.

Example 1: [3] Let $X = R^n$. Let us define the function ||.,…,.|| on X by
$\|x_1, x_2, \ldots, x_n\| = |\det(x_{ij})|$,

$$= abs\left(\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \right)$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ for each $i = 1,2,\ldots,n$. Then $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ is a linear n-normed space.

Example 1: Consider the linear space P_m of real polynomials of degree $\leq m$ on the interval $[0,1]$. Let $x_i \; {}^{nm}_{i=0}$ be $nm+1$ arbitrary but distinct fixed points in $[0,1]$. For $f_1, f_2, \ldots, f_n$ in P_m, let us define
$\|f_1, f_2, \ldots, f_n\| = 0$, if $f_1, f_2, \ldots, f_n$ are linearly dependent,

$$\|f_1, f_2, \ldots, f_n\| = \sum_{i=0}^{nm} |f_1(x_i) f_2(x_i) \ldots f_n(x_i)|,$$

if $f_1, f_2, \ldots, f_n$ are linearly independent.

Then $\|\bullet, \bullet, \ldots, \bullet\|$ is an *n*-norm on P_m.

Solution: If $f_1, f_2, \ldots, f_n$ are linearly dependent, then $\|f_1, f_2, \ldots, f_n\| = 0$. Conversely assume $\sum_{i=0}^{nm} |f_1(x_i) f_2(x_i) \ldots f_n(x_i)| = 0$ This implies that $f_1(x_i) f_2(x_i) \ldots f_n(x_i) = 0$ at $nm+1$ distinct points. Since the degree of each $f_i \le m$, we must have at least one $f_i = 0$. Thus $\|f_1, f_2, \ldots, f_n\| = 0$ if and only if $f_1, f_2, \ldots, f_n$ are linearly dependent. Other properties of *n*-norm can be verified easily.

A sequence $\{x_k\}$ in a linear *n*-normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is called convergent to x if $\lim_{k \to \infty} \|x_k - x, w_2, w_3, \ldots, w_n\| = 0$ for all $w_2, w_3, \ldots, w_n \in X$. It is denoted by $x_k \to x$ as $k \to \infty$

A sequence $\{x_k\}$ in a linear *n*-normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is called Cauchy sequence if $\lim_{k,m \to \infty} \|x_k - x_m, w_2, w_3, \ldots, w_n\| = 0$ for all $w_2, w_3, \ldots, w_n \in X$.

A linear *n*-normed space X is said to be complete if every Cauchy sequence in X is convergent. A complete *n*-normed space is said to be an *n*-Banach space.

SOME ELEMENTARY PROPERTIES

Proposition 1: Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be a linear *n*-normed space.

(*i*) If $\{x_k\}$ is a Cauchy sequence in X, then $\|x_k, w_2, w_3, \ldots, w_n\| : w_2, w_3, \ldots, w_n \in X, k \in N$ is a Cauchy sequence of non-negative reals.

(*ii*) If $\{x_k\}$ and $\{y_k\}$ are Cauchy sequences in X and $\{\alpha_k\}$ is a Cauchy sequence of reals then $\{x_k + y_k\}$ and $\{\alpha_k x_k\}$ are Cauchy sequences in X.

Proof: (*i*) Let $\{x_k\}$ is a Cauchy sequence in X. Then $\lim_{k,m \to \infty} \|x_k - x_m, w_2, w_3, \ldots, w_n\| = 0$, for all $w_2, w_3, \ldots, w_n \in X$. We have

$\|x_k, w_2, w_3, \ldots, w_n\|$

$= \|(x_k - x_m) + x_m, w_2, w_3, \ldots, w_n\|$

$\le \|x_k - x_m, w_2, w_3, \ldots, w_n\| + \|x_m, w_2, w_3, \ldots, w_n\|$

(by nN_4) $\|x_k, w_2, w_3, \ldots, w_n\|$ - $\|x_m, w_2, w_3, \ldots, w_n\| \le \|x_k - x_m, w_2, w_3, \ldots, w_n\|$

Similarly, we have

$\|x_m, w_2, w_3, \ldots, w_n\|$ - $\|x_k, w_2, w_3, \ldots, w_n\|$

$\le \|x_k - x_m, w_2, w_3, \ldots, w_n\|$

Combining the above inequalities, we have

$\big| \|x_k, w_2, w_3, \ldots, w_n\| - \|x_m, w_2, w_3, \ldots, w_n\| \big|$

$\le \|x_k - x_m, w_2, w_3, \ldots, w_n\| \to 0$ as $k, m \to \infty$.

Therefore,

$\big| \|x_k, w_2, w_3, \ldots, w_n\| - \|x_m, w_2, w_3, \ldots, w_n\| \big| \to 0$ as $k, m \to \infty$.

Hence, $\|x_k, w_2, w_3, \ldots, w_n\| : w_2, w_3, \ldots, w_n \in X$, is a Cauchy sequence of non-negative reals.

(*ii*) Let $\{x_k\}$ and $\{y_k\}$ be two Cauchy sequences in X.

Then,
$\lim_{k,m\to\infty}\|x_k - x_m, w_2, w_3,..., w_n\| = 0$, for all $w_2, w_3,..., w_n \in X$ and
$\lim_{k,m\to\infty}\|y_k - y_m, w_2, w_3,..., w_n\| = 0$, for all $w_2, w_3,..., w_n \in X$.

Now,
$$\|(x_k + y_k) - (x_m + y_m), w_2, w_3,..., w_n\|$$
$$= \|(x_k - x_m) + (y_k - y_m), w_2, w_3,..., w_n\|$$
$$\leq \|x_k - x_m, w_2, w_3,..., w_n\| + \|y_k - y_m, w_2, w_3,..., w_n\|$$
$\to 0$ as $k, m \to \infty$.

Hence, $\{x_k + y_k\}$ is a Cauchy sequence in X.

Let $\{\alpha_k\}$ be a Cauchy sequence of reals. Also from (*i*), we have
$\|x_k, w_2, w_3,..., w_n\| : w_2, w_3,..., w_n \in X$ is a Cauchy sequences of reals. Hence they are bounded. We can find K_1, $K_2 \geq 0$ such that $|\alpha_k| \leq K_1$ and $\|x_k, w_2, w_3,..., w_n\| \leq K_2$ for all $k \in N$.

We have $\|\alpha_k x_k - \alpha_m x_m, w_2, w_3,..., w_n\|$
$$= \|\alpha_k x_k - \alpha_k x_m + \alpha_k x_m - \alpha_m x_m, w_2, w_3,..., w_n\|$$
$$\leq \|\alpha_k x_k - \alpha_k x_m, w_2, w_3,..., w_n\|$$
$$+ \|\alpha_k x_m - \alpha_m x_m, w_2, w_3,..., w_n\|$$
$$= |\alpha_k| \ \|x_k - x_m, w_2, w_3,..., w_n\| + |\alpha_k - \alpha_m|$$
$$\|x_m, w_2, w_3,..., w_n\|$$
$$\leq K_1 \ \|x_k - x_m, w_2, w_3,..., w_n\| + K_2 \ |\alpha_k - \alpha_m|$$
$\to 0$, as $k, m \to \infty$.

Thus, $\|\alpha_k x_k - \alpha_m x_m, w_2, w_3,..., w_n\| \to 0$ as $k, m \to \infty$ Hence $\{\alpha_k x_k\}$ is a Cauchy sequence in X.

Proposition 2: In any linear *n*-normed space $(X, \|\bullet,\bullet,...,\bullet\|)$, we have the following

(*i*) If $x_k \to x$ and $y_k \to y$ as $k \to \infty$ then $x_k + y_k \to x + y$ as $k \to \infty$

(*ii*) If $x_k \to x$ and $\alpha_k \to \alpha$ as $k \to \infty$ then $\alpha_k x_k \to \alpha x$ as $k \to \infty$

(*iii*) If $\dim X \geq n$ and $x_k \to x$, $y_k \to y$ as $k \to \infty$ then $x = y$

Proof : (*i*) we have
$$\|(x_k + y_k) - (x + y), w_2, w_3,..., w_n\|$$
$$= \|(x_k - x) + (y_k - y), w_2, w_3,..., w_n\|$$
$$\leq \|x_k - x, w_2, w_3,..., w_n\| + \|y_k - y, w_2, w_3,..., w_n\|$$
$\to 0$ as $k \to \infty$

Therefore, $x_k + y_k \to x + y$ as $k \to \infty$

(*ii*) Using the fact that a real convergent sequence is bounded, we have:

$$\|\alpha_k x_k - \alpha x, w_2, w_3,..., w_n\|$$
$$= \|\alpha_k x_k - \alpha_k x + \alpha_k x - \alpha x, w_2, w_3,..., w_n\|$$
$$\leq \|\alpha_k x_k - \alpha_k x, w_2, w_3,..., w_n\| + \|\alpha_k x - \alpha x, w_2, w_3,..., w_n\|$$
$$= |\alpha_k| \ \|x_k - x, w_2, w_3,..., w_n\| + |\alpha_k - \alpha|$$
$$\|x, w_2, w_3,..., w_n\|$$
$$\leq K \ \|x_k - x, w_2, w_3,..., w_n\| + |\alpha_k - \alpha|$$
$\|x, w_2, w_3,..., w_n\|$ for some $K \geq 0$.

Therefore, $\alpha_k x_k \to \alpha x$ as $k \to \infty$, since $\lim_{k\to\infty}\|x_k - x, w_2, w_3,..., w_n\| = 0$, $\lim_{k\to\infty}|\alpha_k - \alpha| = 0$ and $\|x, w_2, w_3,..., w_n\|$ is finite.

(*iii*) We can write for each $k \in N$ and $w_2, w_3,..., w_n \in X$, $\|x - y, w_2, w_3,..., w_n\|$
$$= \|x - x_k + x_k - y, w_2, w_3,..., w_n\|$$
$$\leq \|x - x_k, w_2, w_3,..., w_n\| + \|x_k - y, w_2, w_3,..., w_n\|$$

It follows that:
$\|x - y, w_2, w_3,..., w_n\| = 0$ for all $w_2, w_3,..., w_n \in X$, since $x_k \to x$, $y_k \to y$ as $k \to \infty$. Hence $x - y, w_2, w_3,..., w_n$ are linearly dependent for all $w_2, w_3,..., w_n \in X$. Since

$\dim X \geq n$, the only way that $x-y$ can be linearly dependent with all vectors $w_2, w_3, ..., w_n \in X$ is for $x-y=0 \Rightarrow x=y$.

Proposition 3: Let $(X, \|\bullet,\bullet,...,\bullet\|)$ be a linear *n*-normed space. If x_k be a Cauchy sequence in *X*, then, $\|x_k - x, w_2, w_3, ..., w_n\| : w_2, w_3, ..., w_n \in X, k \in N$ is a Cauchy sequence of non-negative reals for each $x \in X$.

Proof: We have $\|x_k - x, w_2, w_3, ..., w_n\|$ =

$$\|x_k - x_m + x_m - x, w_2, w_3, ..., w_n\|$$

$$\leq \|x_k - x_m, w_2, w_3, ..., w_n\| + \|x_m - x, w_2, w_3, ..., w_n\|$$

$$\|x_k - x, w_2, w_3, ..., w_n\| - \|x_m - x, w_2, w_3, ..., w_n\|$$

$$\leq \|x_k - x_m, w_2, w_3, ..., w_n\|.$$

Similarly, we have,

$$\|x_m - x, w_2, w_3, ..., w_n\| - \|x_k - x, w_2, w_3, ..., w_n\|$$

$$\leq \|x_k - x_m, w_2, w_3, ..., w_n\|$$

Combining the above inequalities, we have,

$$\left| \|x_k - x, w_2, w_3, ..., w_n\| - \|x_m - x, w_2, w_3, ..., w_n\| \right|$$

$$\leq \|x_k - x_m, w_2, w_3, ..., w_n\|$$

$$\left| \|x_k - x, w_2, w_3, ..., w_n\| - \|x_m - x, w_2, w_3, ..., w_n\| \right|$$

as $k, m \to \infty$, since $\{x_k\}$ is a Cauchy sequence.

Hence, $\|x_k - x, w_2, w_3, ..., w_n\| : w_2, w_3, ..., w_n \in X$ is Cauchy sequence of non-negative reals for each $x \in X$.

Proposition 4: If $\lim_{k\to\infty} \|x_k - x, w_2, w_3, ..., w_n\|$ = 0 then $\lim_{k\to\infty} \|x_k, w_2, w_3, ..., w_n\|$ = $\|x, w_2, w_3, ..., w_n\|$.

Proof: Let $\lim_{k\to\infty} \|x_k - x, w_2, w_3, ..., w_n\| = 0$.

We have,

$$\left| \|x_k, w_2, w_3, ..., w_n\| - \|x, w_2, w_3, ..., w_n\| \right|$$

$$\leq \|x_k - x, w_2, w_3, ..., w_n\|$$

It follows that,

$$\left| \|x_k, w_2, w_3, ..., w_n\| - \|x, w_2, w_3, ..., w_n\| \right| \to 0$$

as $k \to \infty$

Hence,

$$\lim_{k\to\infty} \|x_k, w_2, w_3, ..., w_n\| = \|x, w_2, w_3, ..., w_n\|$$

Proof of the following two Propositions is easy, so omitted. For some similar results on *n*-inner product spaces, one may refer to Hendra Gunawan [2].

Proposition 5: *Limit of every convergent sequence in an n-normed space is unique*.

Proposition 6: *Every convergent sequence in an n-normed space is a Cauchy sequence*.

Now we are ready to give the main Theorem of this paper.

MAIN RESULTS

In this section we prove necessary and sufficient conditions for *n*-norms to be equivalent on linear *n*-normed spaces.

Definition 1: Two *n*-norms $\|\bullet,\bullet,...,\bullet\|_1$ and $\|\bullet,\bullet,...,\bullet\|_2$ on a linear *n*-normed space X are said to be equivalent if there exists constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\|a, w_2, ..., w_n\|_1 \leq \|a, w_2, ..., w_n\|_2 \leq \beta\|a, w_2, ..., w_n\|_1$$

$\forall\ a, w_2, ..., w_n \in X$.

Theorem 1: Two *n*-norms $\|\bullet,\bullet,...,\bullet\|_1$ and $\|\bullet,\bullet,...,\bullet\|_2$ are equivalent on a linear *n*-normed space if and only if every Cauchy sequence with respect to one of the *n*-norms is a Cauchy sequence with respect to other *n*-norm.

Proof: Suppose that two n-norms $\|\bullet,\bullet,\ldots,\bullet\|_1$ and $\|\bullet,\bullet,\ldots,\bullet\|_2$ are equivalent on a linear n-normed space X. Then there exists constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\|a, w_2, \ldots, w_n\|_1 \le \|a, w_2, \ldots, w_n\|_2 \le \beta\|a, w_2, \ldots, w_n\|_1$$

$\forall\ a, w_2, \ldots, w_n \in X$. For a sequence $\{x_k\}$ in X we have

(1) $\alpha\|x_k - x_m, w_2, \ldots, w_n\|_1 \le \|x_k - x_m, w_2, \ldots, w_n\|_2 \le \beta\|x_k - x_m, w_2, \ldots, w_n\|_1$

for all $w_2, w_3, \ldots, w_n \in X$ and $k, m \in N$.

The second inequality shows that if $\{x_k\}$ is Cauchy sequence with respect to $\|\bullet,\bullet,\ldots,\bullet\|_1$ if and only if it is a Cauchy sequence with respect to $\|\bullet,\bullet,\ldots,\bullet\|_2$.

For the converse part, suppose that the n-norms are not equivalent. Then without loss of generality we can assume the following two cases:

(i) we cannot find α such that

$$\alpha\|a, w_2, \ldots, w_n\|_1 \le \|a, w_2, \ldots, w_n\|_2$$

$\forall\ a, w_2, \ldots, w_n \in X$.

(*ii*) we cannot find β such that

$$\|a, w_2, \ldots, w_n\|_2 \le \beta\|a, w_2, \ldots, w_n\|_1$$

$\forall\ a, w_2, \ldots, w_n \in X$.

In case (*i*) for k =1, 2, … , there exists x_k in X such that

(2) $\frac{1}{k}\|x_k, w_2, \ldots, w_n\|_1 > \|x_k, w_2, \ldots, w_n\|_2$

Let $y_k = \frac{1}{\sqrt{k}}\frac{1}{\|x_k, w_2, \ldots, w_n\|_2} x_k$, for each $k \in N$.

Then,

$$\|y_k, w_2, \ldots, w_n\|_2 = \frac{1}{\sqrt{k}} \to 0 \text{ as } k \to \infty$$

and using **(2)** we get

$$\|y_k, w_2, \ldots, w_n\|_1 = \frac{1}{\sqrt{k}}\frac{1}{\|x_k, w_2, \ldots, w_n\|_2}\|x_k, w_2, \ldots, w_n\|_1$$

$$> \frac{k}{\sqrt{k}} = \sqrt{k} \to \infty \text{ as } k \to \infty$$

So, using Proposition 6, $\{y_k\}$ is a Cauchy sequence with respect to $\|\bullet,\bullet,\ldots,\bullet\|_2$ but not with respect to $\|\bullet,\bullet,\ldots,\bullet\|_1$. Similarly, we can prove case (*ii*). Hence the theorem.

Corollary 1: Let $\|\bullet,\bullet,\ldots,\bullet\|_1$ and $\|\bullet,\bullet,\ldots,\bullet\|_2$ be two equivalent n-norms on a linear n-normed space X, then $x_k \to x$ with respect to $\|\bullet,\bullet,\ldots,\bullet\|_1$ if and only if $x_k \to x$ with respect to $\|\bullet,\bullet,\ldots,\bullet\|_2$.

Proof: By replacing '$x_k - x_m$' with '$x_k - x$' in **(1)** of Theorem 1, we get the result.

CONCLUSIONS

The concept of 2-normed spaces was introduced and studied by Siegfried Gähler, a German mathematician who worked at the German Academy of Science, Berlin, in a series of paper in German language published in *Mathematische Nachrichten* in the mid of 1960's. Later, it was further generalized and introduced the notion of n-norm by Misiak.

Very often Gähler has raised the following questions: What is the real motivation for studying 2-norm structure? Is there a physical situation or an abstract concept where norm topology does not work but 2-norm topology does work?

After going through the results of this paper, we see that while studying n-norm structure the main issue should be the use of the meaning of n-norms. We also observe that if a term in the definition of n-norm represents the change of shape, and the n-norm stands for the associated area or center of gravity of the term, maybe we can think of some plausible applicable of the notion of n-norm, and then the generalized convergence make sense. This can also be viewed as: Suppose for a particular output we need n-inputs but with one main input and other (n-1)-inputs are required to complete the process.

REFERENCES

1. Misiak, A. 1989. "n-Inner Product Spaces". *Math. Nachr*. 140:299-319.

2. Gunawan, H. 2002. "On Convergence in n-Inner Product Spaces". *Bull. Malaysian Math. Sc. Soc. (Second Series).* 25:11-16.

3. Gunawan, H. and M. Mashadi. 2001. "On n-normed spaces". *International Journal of Math. & Math. Sci.* 27(10):631-639.

4. Freese, R.W. and Y. Je Cho. 2001. *Geometry of Linear 2-Normed Spaces*. Nova Science Publishers, Inc.: New York, NY.

5. Malceski, R. 2004. "Strong n-Convex n-Normed Space". *Mat. Bull*. 21:81-102.

6. Gahler, S. 1964. "Lineare 2-Normierte Raume". *Math. Nachr.* 28:1-43.

7. Gahler, S. 1969. "Unter Suchungen Uber Veralla gemeinerte m-mertische raume I". *Math. Nachr.*, 40:165-189.

8. Kim, S.S. and Y.J. Cho. 1996. "Strict convexity in Linear n-normed Spaces". *Demonstration Math*. 29(4):739-744.

9. Reddy, B.S. (in press). "Elementary Properties of n-Banach Spaces". *The Journal of the Indian Academy of Mathematics*.

ABOUT THE AUTHORS

Prof. B. Surender Reddy, is an Associate Professor of Mathematics, Department of Mathematics, Osmania University, Hyderabad, Andhra Pradesh, India. He has published/accepted more than 15 papers in different peer-reviewed journals. His research interests are in the areas of functional analysis, operator theory, and its applications.

Prof. Hemen Dutta, is an Assistant Professor of Mathematics, Gauhati University, Kokrajhar Campus, Assam, India. He has published/accepted more than 30 research papers in different peer-reviewed journals. He is a reviewer of different journals and serves as an Associate Editorial Board Member of the International Journal of Open Problems in Computer Science and Mathematics. His research interests are in the areas of mathematical analysis and fuzzy mathematics.

SUGGESTED CITATION

Reddy, B.S. and H. Dutta. 2010. "On equivalence of n-Norms in n-Normed Spaces". *Pacific Journal of Science and Technology*. 11(1):233-238.

[Pacific Journal of Science and Technology](http://www.akamaiuniversity.us/PJST.htm)