Transverse Vibrations of Elastic Thin Beam Resting on Variable Elastic Foundations and Subjected to Traveling Distributed Forces.

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ABSTRACT

In this paper, the problem of the forced vibrations of elastic Bernoulli-Euler beam resting on variable elastic foundations and traversed by uniformly distributed load is investigated. The beam is assumed to be uniform and has simple support at both ends. The moving distributed force is assumed to move with constant velocity. The robust technique called Galerkin's method in conjuction with integral transform method are used to treat the fourth order partial differential equations describing the motion of the beam-load system. Results show that, increases in the values of beam parameters, axial force N, and foundation modulus K, significantly reduce the deflection profile of the vibrating beam. It is equally found that, incorporating axial force N, foundation modulus K, and a damping term into the governing equation of motion increases the critical velocity of dynamical system thereby reducing the risk of resonance.

(Keywords: damping, distributed forces, foundation stiffness, response of structures, moving masses)

INTRODUCTION

This paper is concerned with assessing the problem of the complexity of the interraction of deformable elastic system (beam or plate) and the dynamic subsystem (moving force or moving mass) traversing it. Due to its very wide range of applications in diverse areas, researchers from the field of engineering, mathematical physics, and applied physics have made significant contributions in this area of study in the past few years [1-10].

It is well known that when loads move on structural members that offer resistance to bending, it produces two effects which cause the structure to vibrate continously. These two effects in the field of structural dynamics are termed the moving force effects and moving mass effect [11,12].

In times past, analytical studies concerning the vibrations of elastic structures under the passage of fast travelling loads have been exclusively reserved in the literature for the case of structure-load interractions where the moving load is assumed to be concentrated loads. These types of loads are only useful in mathematical idealizations, but in reality cannot be found in the real world. The more practical cases where the moving load is a distributed type is very seldom found in the literature.

Among the few studies about the response of elastic structures to moving distributed loads known in the literature are the works of Gbadeyan and Dada who studied the dynamic response of plates on pasternak foundation to distributed moving load. In this study the moving force plate model is considered and it is found that an increase in the area of the disrtibution of the moving mass causes a reduction in the maximum dynamic deflection.

In all of the aforementioned studies, the authors neglected the damping term in the governing differential equation of motion and effects of elastic foundation of non- uniform stiffness was not investigated. This study therefore investigates the transverse displacement response of axially prestressed thin beams resting on variable elastic foundation to moving distributed loads.

FORMULATION OF THE BOUNDARY VALUE PROBLEM

Consider a prismatic Bernoulli-Euler beam of lenght L resting on a Winkler foundation and traversed by uniformly distributed load moving at constant velocity. The vibrational behaviour of the beam-load system is described by the fourth order partial differential equation with variable coefficient given as:

$$EJ\frac{\partial^4 w(x,t)}{\partial x^4} - N\frac{\partial^2 w(x,t)}{\partial x^2} + \varepsilon_0 \frac{\partial w(x,t)}{\partial x} + \mu \frac{\partial^2 w(x,t)}{\partial x^2} + K(x)w(x,t) = P(x,t) \quad \textbf{(1)}$$

where: $w(x,t)$ is the lateral deflection of the beam measured from its equilibrum position

EJ is the flexural rigidity of the beam
E is the modulus of elasticity
J is the moment of inertia
K_0 is the coefficient of Winkler foundation
g is the acceleration due to gravity
ε_0 is the fixed lenght of the beam
x is the spatial coodinate
μ is mass per unit lenght of the beam
t is the time

The moving load on the beam under consideration has mass commensurable with the mass of the beam and the load on the beam is assumed to be of mass *m* moving with constant velocity *c*. A variable elastic foundation of the form:

$$K = K_0(4x - 3x^2 + x^2) \quad \textbf{(2)}$$

where K_0 is the foundation constant, is considered.

Furthermore, we assume that the beam has simple supports at the end x=0 and at the end x=L, so that both the bending moment and the deflection vanish at both ends. Thus, the pertinent boundary conditions that pertain at these two ends is given as:

$$W(0,t) = 0 = W(L,t), \qquad \frac{\partial^2}{\partial x^2}W(0,t) = 0 = \frac{\partial^2}{\partial x^2}W(L,t)$$

(3a) (i)

(ii)

and the intial conditions are:

(iii)

$$W(x,0) = 0 = \frac{\partial^2}{\partial t^2}W(x,0) \quad \textbf{(3b)}$$

The traveling time *t* of the moving load is assumed to be limited to that interval of time within which the the moving load is on the beam, that is:

$$0 \le ct \le L \quad \textbf{(4a)}$$

and the moving force $P(x,t)$ is assumed to be a uniformly distributed single point load given as:

$$P(x,t) = P_0 H(x - ct) \quad \textbf{(4b)}$$

where P_0 is the mass of the load multiply the acceleration due to gravity g and $H(x-ct)$ is the heaviside unit step function defined as:

$$H(x-ct) = \begin{cases} 0, & for \;\; x < 0 \\ 1, & for \;\; x > 0 \end{cases} \quad \textbf{(5)}$$

with the properties,

(i) $$\frac{d}{dx}[H(x-ct)] = \delta(x-ct) \quad \textbf{(6)}$$

(ii) $$f(x)H(x-ct) = \begin{cases} 0, & for \;\; x < ct \\ f(x), & for \;\; x \ge ct \end{cases} \quad \textbf{(7)}$$

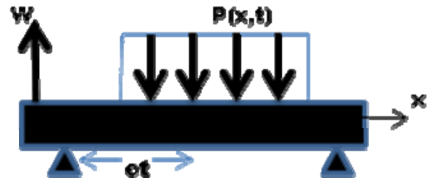

Figure1: A Disrtibuted Load on Elastic Beam.

where $\delta(x-ct)$ represents the Dirac delta function. $H(x-ct)$ is a typical engineering function made to measure engineering applications which often involved functions that are either "off" or "on". Substituting (3), (4), and (5) into (1), one obtains:

$$EJ\frac{\partial^4 w(x,t)}{\partial x^4} - N\frac{\partial^2 w(x,t)}{\partial x^2} + \varepsilon_0\frac{\partial w(x,t)}{\partial x} + \mu\frac{\partial^2 w(x,t)}{\partial x^2} + K = K_0(4x-3x^2+x^2)w(x,t) = MgH(x-ct) \quad \textbf{(8)}$$

SOLUTION PROCEDURES:

In this section, we proceed to solve the above initial-boundary value problem (8) by employing Galerkin method. The method expresses the solutions of the equation (8) respectively as:

$$W_m(x,t) = \sum_{m=1}^{n} Y_m(t)U_m(x) \quad \textbf{(9)}$$

where $U_m(x)$ is given as:

$$U_m(x) = Sin\frac{m\pi x}{L} \quad \textbf{(10)}$$

The function $U_m(x)$ is choosen to satisfy the pertinent boundary conditions. Thus substituting (9) and (10) and its derivatives into (8), we have:

$$EJ\sum_{m=1}^{n}\left(\frac{m\pi}{L}\right)^4 Y_m(t)Sin\frac{m\pi x}{L} + N\sum_{m=1}^{n}\left(\frac{m\pi}{L}\right)^2 Y_m(t)Sin\frac{m\pi x}{L} + \varepsilon_0\sum_{m=1}^{n}\dot{Y}_m(t)Sin\frac{m\pi x}{L}$$
$$+\mu\sum_{m=1}^{n}\ddot{Y}_m(t)Sin\frac{m\pi x}{L} + K_0(4x-3x^2+x^2)\sum_{m=1}^{n}Y_m(t)Sin\frac{m\pi x}{L} = MgH(x-ct) \quad \textbf{(11)}$$

which after some re-arrangements gives:

$$\sum_{m=1}^{n}\left\{EJ\left(\frac{m\pi}{L}\right)^4 Y_m(t)Sin\frac{m\pi x}{L} + N\left(\frac{m\pi}{L}\right)^2 Y_m(t)Sin\frac{m\pi x}{L} + \varepsilon_0\dot{Y}_m(t)Sin\frac{m\pi x}{L} + \mu\ddot{Y}_m(t)Sin\frac{m\pi x}{L}\right.$$
$$\left. +K_0(4x-3x^2+x^2)Y_m(t)Sin\frac{m\pi x}{L} = MgH(x-ct)\right\} \quad \textbf{(12)}$$

In order to determine $Y_m(t)$, it is required that the equation (2.4) be orthogonal to the function $U_k(x)$. Hence:

$$\int_0^L \left[\sum_{m=1}^{n} \left\{ EJ\left(\frac{m\pi}{L}\right)^4 Y_m(t) Sin\frac{m\pi x}{L} + N\left(\frac{m\pi}{L}\right)^2 Y_m(t) Sin\frac{m\pi x}{L} + \varepsilon_0 \dot{Y}_m(t) Sin\frac{m\pi x}{L} + \mu \ddot{Y}_m(t) Sin\frac{m\pi x}{L} \right.\right.$$

$$\left.\left. + K_0(4x-3x^2+x^2)Y_m(t) Sin\frac{m\pi x}{L} = MgH(x-ct) \right\}\right] Sin\frac{k\pi x}{L}dx$$

(13)

Equation (13) can further be re-arranged as:

$$\sum_{m=1}^{n} \ddot{Y}_m(t) + Q_1\dot{Y}_m(t) + Q_2 Y_m(t) = Q_3 \int_0^L H(x-ct) Sin\frac{m\pi x}{L}dx \quad \textbf{(14)}$$

where,

$$Q_1 = \frac{H_1+H_2+H_5}{H_4}, \qquad Q_1 = \frac{H_3}{H_4} \qquad Q_3 = \frac{H_6}{H_4} \quad \textbf{(15)}$$

and
$$\mathrm{H}_1 = \frac{EJ}{\mu}\left(\frac{m\pi}{L}\right)^4 \int_0^L Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \quad \textbf{(16)}$$

$$\mathrm{H}_2 = \frac{N}{\mu}\left(\frac{m\pi}{L}\right)^2 \int_0^L Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \quad \textbf{(17)}$$

$$\mathrm{H}_3 = \frac{\varepsilon_0}{\mu}\int_0^L Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \quad \textbf{(18)}$$

$$\mathrm{H}_4 = \int_0^L Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \quad \textbf{(19)}$$

$$\mathrm{H}_5 = \frac{K_0}{\mu}\int_0^L (4x-3x^2+x^3) Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \quad \textbf{(20)}$$

$$\mathrm{H}_6 = \frac{mg}{\mu} \quad \textbf{(21)}$$

Expressing the Heaviside unit step function as a fourier cosine series in equation (2.5) to (2.7) then we have:

$$H(x-ct) = \int_0^L \left[\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} Cos\frac{n\pi ct}{L} Cos\frac{n\pi x}{L}\right]dx$$

(22)

Thus,

$$H = (x-ct) = \frac{-L}{k\pi} Cos k\pi + \frac{L}{k\pi} Cos\frac{k\pi ct}{L} \quad \textbf{(23)}$$

Considering only the mth particle of the dynamical system, thus, one obtains,

$$\ddot{Y}_m(t) + Q_1\dot{Y}_m(t) + Q_2Y_m(t) = Q_3\left[\frac{-L}{k\pi}Cosk\pi + \frac{L}{k\pi}Cos\frac{k\pi ct}{L}\right] \tag{24}$$

To obtain solution to the second ordinary differential equation (24) above, we subjected it to a Laplace transform defined as:

$$(\tilde{\cdot}) = \int_0^\infty (\cdot)e^{-st\,dt} \tag{25}$$

Thus,

$$Y(s) = \frac{Q_3\left[\frac{-L}{k\pi}Cosk\pi\left(\frac{1}{S}\right) + \frac{L}{k\pi}\left(\frac{S}{S^2+\left(\frac{k\pi}{L}\right)^2}\right)\right]}{S^2 + Q_1S + Q_2} \tag{26}$$

which reduces to:

$$Y(s) = Q_3\left[\frac{a}{s} + b\left(\frac{s}{s^2+\eta^2}\right)\right]\cdot\frac{1}{(s-\alpha_1)(s-\alpha_2)} \tag{27}$$

where,

$$a = \frac{-L}{k\pi}Cosk\pi, \qquad b = \frac{L}{k\pi} \quad \textbf{and} \quad \alpha_1 = \frac{-Q_1 + \sqrt{Q_1^2 - 4Q_2}}{2}, \quad \alpha_2 = \frac{-Q_1 - \sqrt{Q_1^2 - 4Q_2}}{2} \tag{28}$$

Further simplification yields:

$$Y(s) = \frac{Q_3}{\alpha_1 - \alpha_2}\left\{\frac{a}{s}\cdot\frac{1}{s-\alpha_1} - \frac{a}{s}\cdot\frac{1}{s-\alpha_1} + b\frac{s}{s^2+\eta^2}\cdot\frac{1}{s-\alpha_2} - b\frac{s}{s^2+\eta^2}\cdot\frac{1}{s-\alpha_2}\right\} \tag{29}$$

In order to obtain the Laplace inversion of (29), we make the following representations:

$$\left.\begin{array}{l} g_1(t) = a, \quad g_2(t) = Cos\eta t, \\ f_1(t) = e^{\alpha_2 t}, \; f_2(t) = e^{\alpha_1 t} \end{array}\right\} \tag{30}$$

so that the Laplace inversion of (29) is the convolution of *fi*'s and *gi*'s defined by:

$$f_i * g_i = \int_0^t f_i(t-u)g_i(u)du \qquad r = 1,2,3...... \tag{31}$$

Thus the Laplace inversion of equation (29) is given by:

$$U_j(t) = \frac{Q_3}{\alpha_1 - \alpha_2}\left\{-\frac{a}{\alpha_2} + \frac{a}{\alpha_1} + b\left[\frac{\eta Sin\eta t - \alpha_2 Cos\eta t}{\alpha_2^2 - \eta^2}\right] - b\left[\frac{\eta Sin\eta t - \alpha_1 Cos\eta t}{\alpha_1^2 - \eta^2} - \right]\right\} \quad \textbf{(32)}$$

where,

$$I_A = ae^{\alpha_2 t}\int_0^t e^{\alpha_2 u} du \quad \textbf{(33)}$$

$$I_B = ae^{\alpha_1 t}\int_0^t e^{\alpha_1 u} du \quad \textbf{(34)}$$

$$I_C = ae^{\alpha_2 t}\int_0^t e^{-\alpha_2 u} Cos\eta u du \quad \textbf{(35)}$$

$$I_D = ae^{\alpha_1 t}\int_0^t e^{-\alpha_1 u} Cos\eta u du \quad \textbf{(36)}$$

Using (33) to (36) in (32), one obtains,

$$U_j(t) = \frac{Q_3}{\alpha_1 - \alpha_2}\left\{a\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right) + b\left[\frac{\eta Sin\eta t - \alpha_2 Cos\eta t}{\alpha_2^2 - \eta^2}\right] - b\left[\frac{\eta Sin\eta t - \alpha_1 Cos\eta t}{\alpha_1^2 - \eta^2} - \right]\right\} Sin\frac{m\pi x}{L} \quad \textbf{(37)}$$

which on inversion yields:

$$w_m(x,t) = \frac{Q_3}{\alpha_1 - \alpha_2}\sum_{m=1}^{n}\left\{a\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right) + b\left[\frac{\eta Sin\eta t - \alpha_2 Cos\eta t}{\alpha_2^2 - \eta^2}\right] - b\left[\frac{\eta Sin\eta t - \alpha_1 Cos\eta t}{\alpha_1^2 - \eta^2} - \right]\right\} Sin\frac{m\pi x}{L}$$

(38)

NUMERICAL RESULTS AND DISCUSSION

For the purpose of Numerical analysis of our dynamical system, the uniform thin beamof lenght 12.192m is considered.

Also $\frac{EI}{\mu} = 2200m^4/s^2$, speed of the mass is 8.128m/s and the ratio of the mass of the load to the beam is 0.2.

The transverse deflection of the beam are calculated and plotted against the time for various values of axial force N and subgrade K. Values of N between 0 and 2000000 were used while the values of K were varied between 0 N/m^3 .

The results are as shown on the various graphs below. Figure 2 displays transverse displacement response of a simply supported uniform beam under the action of distributed forces moving at a variable velocities for various values of axial force N for fixed values of foundation moduli K=40,000.

The figure shows that as N increases, the deflection of the uniform beam decreases. In a similar way, for various time t, the deflection profile of the beam for various values of foundation moduli K and for fixed axial force N are shown in Figure 3. It is observed that higher values of foundation moduli reduce the deflection profile of the beam.

CONCLUSION

In this paper the dynamic behavior of finite elastic beam resting on elastic foundation and subjected to moving distributed forces is investigated.

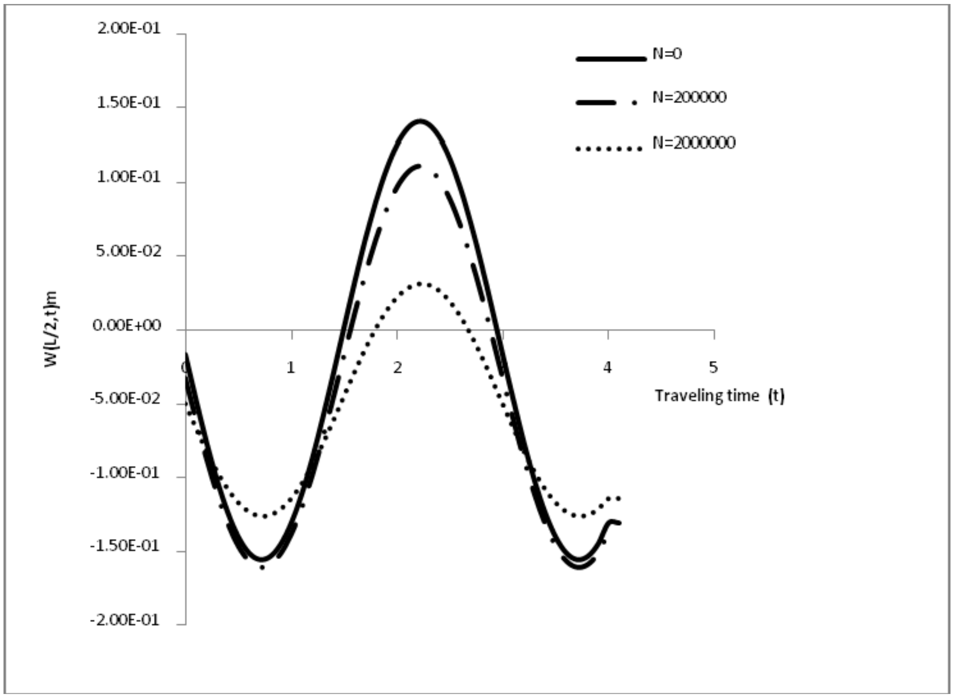

Figure 1: The Deflection Profile of the Simply Supported Thin Beam Under the Action of Constant Velocity for Various Values of Axial Force N and for Fixed Value of Foundation Modulus K (400).

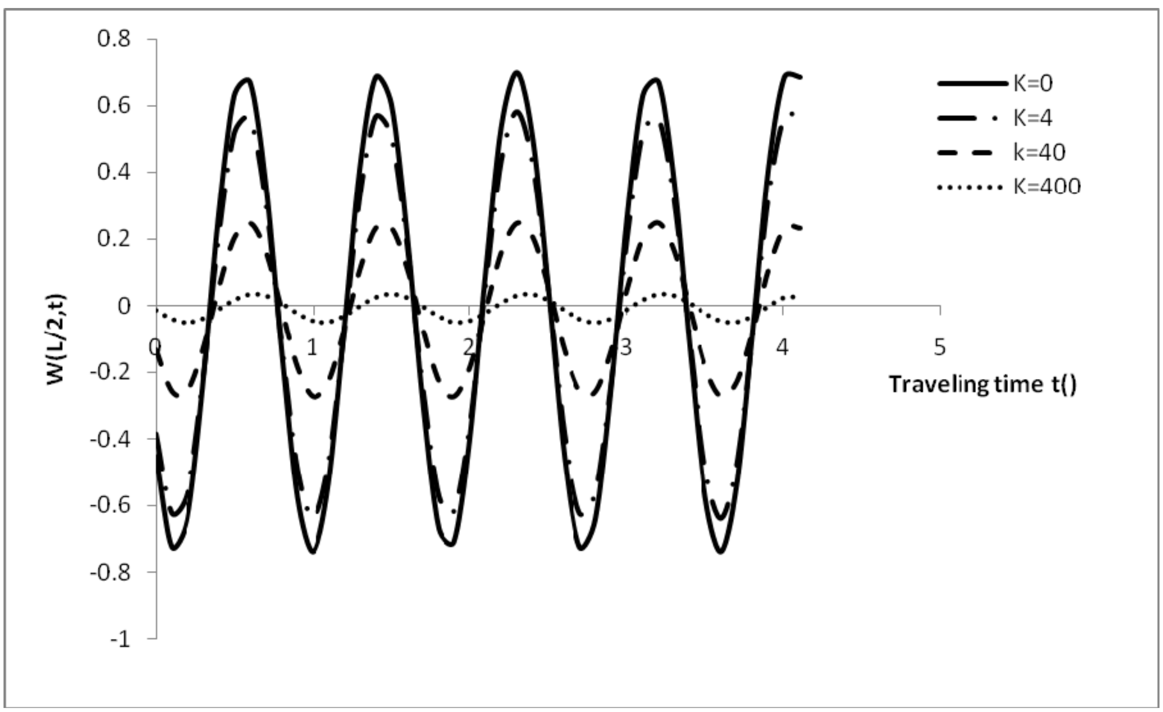

Figure 2: The Deflection Profile of the Simply Supported Thin Beam Under the Action of Constant Velocity for Various Values of Foundation Moduli K for Fixed Value of Axial Force N (20000).

The bean is assumed to be under tensile stress and the motion of the moving load is assumed to be a constant velocity type of motion. Spectral Galerkin's method in conjunction with integral transform method is used to obtain a closed form solution to this dynamical problem.

The results show that with Increase in the values of beam parameters such as axial force and foundation stiffness, the response amplitude of the vibrating beam reduces. Furthermore, Increase in the values of these parameters and the damping coefficient produces a significant effect on the critical velocity of the beam-load system and the risk of resonance is sufficiently reduced. This result is in perfect agreement with existing result [11, 12].

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