Some New Type of Generalized Difference Paranormed Sequence Spaces Defined by Sequence of Orlicz Functions Associated with Multiplier Sequences.

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ABSTRACT
In this article we introduce some new type of sequence spaces
\[ c\{M, \Delta^n, \Lambda, p\} \]
\[ c_0\{M, \Delta^n, \Lambda, p\} \]
and \[ \ell_\infty\{M, \Delta^n, \Lambda, p\} \].

Also studied some different properties and established some inclusion relations.

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INTRODUCTION
Throughout \( w, c, c_0, \ell_\infty \) denote the spaces of all, convergent, null, and bounded sequences, respectively.

The notion of difference sequence space was introduced by Kizmaz [5] as follows:

\[ X(\Delta) = \{(x_k) \in w: (\Delta x_k) \in X\} \]

for \( X = \ell_\infty, c, c_0 \) where \( n \in N \); \( \Delta^0 x_k = x_k \) for all \( k \in N \), and

\[ \Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1} = \sum_{v=0}^{n} (-1)^v \left(\begin{array}{c} n \\ v \end{array}\right) x_{k+v} \]

for all \( k \in N \).

The notion of paranormed sequence spaces was introduced by Nakano [11] and Simons [14]. It was further investigated by Maddox [10], Lascarides [7] and many others.

We write \( r = (r_k) = \left(\frac{1}{p_k}\right) \).

An Orlicz function is a function \( M: [0, \infty) \rightarrow [0, \infty) \), which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0 \) for \( x > 0 \) and \( M(x) \rightarrow \infty \) as \( x \rightarrow \infty \).

The study of Orlicz sequence spaces was initiated with certain specific purpose in Banach space theory. Indeed Lindberg [9] got interested in Orlicz spaces in connection with finding Banach spaces with symmetric. Subsequently Lindenstrauss and Tzafriri [8] investigated these Orlicz sequence spaces in detail. Later on, different classes of sequence spaces defined by Orlicz function were studied by Nung and Lee [12], Woo [16], Parashar and Choudhary [13] and many others. The Orlicz sequence spaces are the special case of Orlicz spaces studied by [6].

The scope for the studies on sequence spaces was extended by using the notion of an associated multiplier sequence. Goes and Goes [3], defined the differentiated sequence space \( dE \) and integrated sequence space \( \int E \) for a given
sequence space $E$, by using the multiplier sequences $(k^i)$ and $(k)$ respectively. Tripathy [15] used a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars for his studies on sequence spaces, associated with multiplier sequences.

In this paper we shall consider a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars. Then for a sequence space $E$, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence $\Lambda$ is defined as:

$$E(\Lambda) = \{(x_k) \in w: (\lambda_k x_k) \in E\}.$$ 

DEFINITIONS AND PRELIMINARIES

**DEFINITION.** A sequence space $E$ is said to be solid (or normal) if $(x_k) \in E$ and $(\alpha_k)$ a sequence of scalars with $|\alpha_k| \leq 1$, for all $k \in N$, then $(\alpha_k x_k) \in E$.

**DEFINITION.** A sequence space $E$ is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$, where $\pi$ is a permutation of $N$.

Let $M = (M_k)$ be a sequence of Orlicz functions. We introduce the following sequence spaces associated with a multiplier sequence $\Lambda = (\lambda_k)$ is a sequence of non-zero scalars:

$$c(M, \Delta^m, \Lambda, p) = \{(x_k) \in w: \lim_{k \to \infty} \left[ M_k\left(\frac{\lambda_k (x_k)}{p}\right)^p \right] r_k = 0, \text{ for some } p > 0 \};$$

$$c_0(M, \Delta^m, \Lambda, p) = \{(x_k) \in w: \lim_{k \to \infty} \left[ M_k\left(\frac{\lambda_k (x_k)}{p}\right)^p \right] r_k = 0, \text{ for some } L \in C \text{ and } p > 0 \};$$

$$\ell_\infty(M, \Delta^m, \Lambda, p) = \{(x_k) \in w: \sup_k \left[ M_k\left(\frac{\lambda_k (x_k)}{p}\right)^p \right] r_k < \infty, \text{ for some } p > 0 \};$$

**DISCUSSION**

(i) Let $M_k(x) = x$, for all $k$ in $N$ and $m = 1$, then the spaces $c(\Delta, \Lambda, p)$, $c_0(\Delta, \Lambda, p)$ and $\ell_\infty(\Delta, \Lambda, p)$ introduced and studied by Tripathy [15].

(ii) Let $M_k(x) = x$ and $\lambda_k = 1$, for all $k$ in $N$ and $m = 0$, then the spaces $c_0(p)$, $c(p)$ and $\ell_\infty(p)$ are studied by Lascarides [7].

(iii) Let $M_k(x) = x$, $\lambda_k = 1$, $p_k = 1$, for all $k$ in $N$ and $m = 0$, then $c$, $c_0$ and $\ell_\infty$ are the spaces of convergent, null and bounded sequences respectively.

The following results will be used for establishing some results of this article.

**LEMMA 1 (Lascarides [7], Proposition 1).**

Let $h = \inf_k p_k$, $H = \sup_k p_k$. Then the following conditions are equivalent:

(i) $h < \infty$ and $h > 0$;

(ii) $c_0(p) = c_0$ or $\ell_\infty(p) = \ell_\infty$;

(iii) $\ell_\infty(p) = \ell_\infty(p)$;

(iv) $c_0(p) = c_0(p)$;

(v) $\ell(p) = \ell(p)$.

**LEMMA 2 (Lascarides [7], Corollary 1).**

Let $p, q$ be two sequences of strictly positive numbers. Then $c_0(p) \cong c_0(q)$ if and only if there exists a sequence $u = (u_k)$ of strictly positive numbers such that

$$\lim_q \lim_{k \to \infty} \sup_k \left[ u_k p_k^{q_k^{-1}} \eta_k^{-\left(\frac{\eta_k}{p_k}\right)^{q_k}} \right] = 0$$

and

$$\lim_q \lim_{k \to \infty} \sup_k \left[ u_k q_k^{p_k^{-1}} \eta_k^{\left(\frac{\eta_k}{q_k}\right)^{p_k}} \right] = 0$$

**LEMMA 3 (Lascarides [7], Corollary 2).**

Let the sequence $a = (a_k) = \left( q_k^{p_k^{-1}} p_k^{q_k^{-1}} \right)$. Then $c_0(p) \cong c_0(q)(a)$ if and only if the following conditions hold:

$$\lim_q \lim_{k \to \infty} \sup_k \eta_k^{-\left(\frac{\eta_k}{p_k}\right)^{q_k}} = 0$$

and
\begin{align}
\lim_{\eta} \limsup_{k} \eta^{\frac{1}{\eta}-\frac{1}{\eta+1}} &= 0 \tag{4} \\
\text{LEMMA 4 (Lascarides\cite{7}, Corollary 3).} \\
\text{Let the sequence } a = (a_k) = \left( q^{\frac{1}{q}} p^{-\frac{1}{q}} \right). \text{ Then } \lim_{k \to \infty} \left( \frac{1}{q_k} - \frac{1}{q_k + 1} \right) = 0 \text{ implies } c_0(p) \cong c_0(q)(a). \\
\text{LEMMA 5 (Lascarides\cite{7}, Proposition 3).} \\
\text{Let } f_k = \frac{p_k}{q_k}, \text{ for every } k \in \mathbb{N}. \text{ Let } (f_k) \text{ and } (f_k^{-1}) \text{ both be in } \ell_{\infty}. \text{ Then } \ell_{\infty}(p) \cong \ell_{\infty}(q)(f).
\end{align}

**MAIN RESULTS**

**THEOREM 1.** For any sequence \( p = (p_k) \) of strictly positive real numbers, the classes \( c(M, \Delta^m, \Lambda, p), c_{0}(M, \Delta^m, \Lambda, p) \) and \( \ell_{\infty}(M, \Delta^m, \Lambda, p) \) are linear spaces.

**PROOF.** We prove the result for the case \( c(M, \Delta^m, \Lambda, p) \). The rest of the results follow similarly. Let \( (x_k), (y_k) \in c(M, \Delta^m, \Lambda, p) \) and \( \alpha, \beta \) be two scalars in \( C \). Then there exist \( \rho_1 > 0 \) and \( \rho_2 > 0 \) such that:

\[
\lim_{k \to \infty} \left\{ M_k \left( \frac{\lambda_k \left( \Delta^m x_k - L_k \right)}{\rho_1} \right)^{p_k} r_k \right\} = 0, \text{ for some } L_1 \in C, \\
\text{and } \lim_{k \to \infty} \left\{ M_k \left( \frac{\lambda_k \left( \Delta^m y_k - L_k \right)}{\rho_2} \right)^{p_k} r_k \right\} = 0, \text{ for some } L_2 \in C.
\]

Let \( \rho = \max \{2|\alpha| \rho_1, 2|\beta| \rho_2 \} \). Then we have:

\[
\left\{ M_k \left( \frac{\lambda_k \left( \Delta^m x_k + \beta \Delta^m y_k \right) - (a_L + b_L)}{\rho} \right)^{p_k} r_k \right\} \leq D \left\{ M_k \left( \frac{\lambda_k \left( \Delta^m x_k - L_k \right)}{\rho_1} \right)^{p_k} r_k \right\} + D \left\{ M_k \left( \frac{\lambda_k \left( \Delta^m y_k - L_k \right)}{\rho_2} \right)^{p_k} r_k \right\}, \text{ where } D = \max(1, 2^{H+1}) \text{ and } H = \sup_k p_k < \infty.
\]

Taking limit \( k \to \infty \) on both sides of the above inequality, we have:

\[
(\alpha(x_k) + \beta(y_k)) \in c(M, \Delta^m, \Lambda, p).
\]

Therefore \( c(M, \Delta^m, \Lambda, p) \) is a linear space.

**THEOREM 2.** Let \( 0 < \inf p_k \leq \sup p_k < \infty \), then the space \( \ell_{\infty}(M, \Delta^m, \Lambda, p) \) is a paranormed space, paranormed by

\[
f_\Delta(x) = \sum_{x_{i} \neq \infty} x_{i} + \inf \left\{ \rho^\Delta \sup_{k \geq 1} \left( M_k \left( \frac{\lambda_k \left( \Delta^m x_k \right)}{\rho} \right)^{p_k} r_k \right) \leq 1, \text{ for some } \rho > 0 \right\},
\]

where \( K = \max(1, H) \).

**PROOF.** Clearly \( f_\Delta(x) = f_\Delta(-x) \). Since \( M_k(0) = 0 \), for all \( k \in N \). We get \( f_\Delta(\overline{0}) = 0 \), for \( x = \overline{0} \).
Let \( x = (x_k), \ y = (y_k) \) be two elements of \( \ell_{\infty}(\mathbb{M}, \Delta^m, \Lambda, \rho) \) and let us choose \( \rho_1 > 0 \) and \( \rho_2 > 0 \) such that:

\[
\sup_{k \geq 1} \left\{ M_k \left( \frac{|x_k|}{\rho_1} \right) \right\}^{\frac{1}{p_k}} \leq 1
\]

and

\[
\sup_{k \geq 1} \left\{ M_k \left( \frac{|y_k|}{\rho_2} \right) \right\}^{\frac{1}{p_k}} \leq 1
\]

Let \( \rho = \rho_1 + \rho_2 \). Then we get:

\[
\sup_{k \geq 1} \left\{ M_k \left( \frac{|x_k + y_k|}{\rho} \right) \right\}^{\frac{1}{p_k}} \leq 1
\]

Now,

\[
f_{\Delta}(x + y) = \sum_{k = 1}^{m} |x_k + y_k| + \inf \left\{ (\rho_1 + \rho_2) \sum_{k = 1}^{m} \left\{ M_k \left( \frac{|x_k|}{\rho_1 + \rho_2} \right) \right\}^{\frac{1}{p_k}} \right\} \leq 1
\]

Finally let \( \gamma \) be a given scalar in \( \mathbb{C} \), then the continuity of the scalar multiplication follows from the following equality:

\[
f_{\Delta}(\gamma x) = \sum_{k = 1}^{m} |\gamma x_k| + \inf \left\{ (\rho) \sum_{k = 1}^{m} \left\{ M_k \left( \frac{|x_k|}{\rho} \right) \right\}^{\frac{1}{p_k}} \right\} \leq 1, \text{ for some } \rho > 0
\]

COROLLARY 1. Let \( p \in \ell_{\infty} \), then the spaces \( c(\mathbb{M}, \Delta^m, \Lambda, \rho) \) and \( c_0(\mathbb{M}, \Delta^m, \Lambda, \rho) \) are paranormed spaces, paranormed by \( f_{\Delta} \) defined above.

THEOREM 3. Let \( p \in \ell_{\infty} \), then the spaces \( c(\mathbb{M}, \Delta^m, \Lambda, \rho), c_0(\mathbb{M}, \Delta^m, \Lambda, \rho) \) and \( \ell_{\infty}(\mathbb{M}, \Delta^m, \Lambda, \rho) \) (with inf \( p_k \geq 0 \)) are complete paranormed spaces, paranormed by \( f_{\Delta} \).

PROOF. Let \( (x^n) \in \ell_{\infty}(\mathbb{M}, \Delta^m, \Lambda, \rho) \) be a Cauchy sequence, where \( x^n = (x^n_k)_{k=1}^{\infty} \), for all \( n \in \mathbb{N} \). Then, we have,

\[
f_{\Delta}(x^i - x^j) \to 0, \text{ as } i, j \to \infty.
\]
For a given \( \epsilon > 0 \), let \( u \) and \( v_0 \) be such that \( \frac{\epsilon}{u v_0} > 0 \) and \( M_k \left( \frac{u v_0}{2} \right) \geq \sup (p_k)^\frac{\epsilon}{2} \).

Then \( f_\Delta (x^i - x^j) \to 0 \), as \( i, j \to \infty \), implies that there exists \( n_0 \in \mathbb{N} \) such that

\[
f_\Delta (x^i - x^j) < \frac{\epsilon}{u v_0}, \text{ for all } i, j \geq n_0
\]

\[
\Rightarrow \sum_{x=1}^{m} |x^i_x - x^j_x| < \frac{\epsilon}{u v_0}
\]

and

\[
\inf \left\{ (\rho) \frac{\epsilon}{n} : \sup_{k=1} \left\{ M_k \left( \frac{\lambda_k (\Delta^m x^i_k - \Delta^m x^j_k) \rho}{\Delta^m x^i_k - \Delta^m x^j_k} \right) \right\} \leq 1, \text{ for } \rho > 0 \right\} < \frac{\epsilon}{u v_0}\]

\[
\Rightarrow (x^i_x) \text{ is a Cauchy sequence in } \mathbb{C}, \text{ so } (x^i_x) \text{ is convergent in } \mathbb{C}. \text{ Let } \lim_{i \to \infty} x^i_x = x^i_x \text{ (say)}.
\]

Then

\[
\lim_{i \to \infty} \sum_{x=1}^{m} |x^i_x - x^j_x| < \frac{\epsilon}{u v_0}
\]

\[
\Rightarrow \sum_{x=1}^{m} |x^i_x - x^j_x| < \frac{\epsilon}{u v_0}.
\]

From (6), we have,

\[
\left\{ M_k \left( \frac{\lambda_k (\Delta^m x^i_k - \Delta^m x^j_k) \rho}{\Delta^m x^i_k - \Delta^m x^j_k} \right) \right\} \leq 1
\]

\[
\Rightarrow M_k \left( \frac{\lambda_k (\Delta^m x^i_k - \Delta^m x^j_k) \rho}{\Delta^m x^i_k - \Delta^m x^j_k} \right) \leq \left( \frac{\epsilon}{2} \right) \frac{\epsilon}{\epsilon} \leq M_k \left( \frac{u v_0}{2} \right)
\]

\[
\Rightarrow |\Delta^m x^i_k - \Delta^m x^j_k| \leq \frac{\epsilon}{u v_0} \cdot \frac{\epsilon}{\epsilon} < \frac{\epsilon}{2}
\]

\[
\Rightarrow (\lambda_k (\Delta^m x^i_k) ) \text{ is a Cauchy sequence in } \mathbb{C}, \text{ for all } k \in \mathbb{N}.
\]

Hence \( (\lambda_k (\Delta^m x^i_k) ) \) converges in \( \mathbb{C} \). Thus \( \lim_{i \to \infty} \lambda_k (\Delta^m x^i_k) = (\lambda_k (\Delta^m x^i_k) ) \), for all \( k \in \mathbb{N} \).

By the continuity of \( M_k \), we have:

\[
\lim_{j \to \infty} \sup_{k=1} \left\{ M_k \left( \frac{\lambda_k (\Delta^m x^i_k - \Delta^m x^j_k) \rho}{\Delta^m x^i_k - \Delta^m x^j_k} \right) \right\} \leq 1
\]

\[
\Rightarrow \sup_{k=1} \left\{ M_k \left( \frac{\lambda_k (\Delta^m x^i_k - \Delta^m x^j_k) \rho}{\Delta^m x^i_k - \Delta^m x^j_k} \right) \right\} \leq 1.
\]

Let \( i \geq n_0 \), and taking infimum over \( \rho \), we have,

\[
f_\Delta (x^i - x) < \epsilon.
\]

Then \( (x^i - x) \in \ell_{\infty} \{ M, \Delta^m, \Lambda, \rho \} \).

Therefore \( x = x^i - (x^i - x) \in \ell_{\infty} \{ M, \Delta^m, \Lambda, \rho \} \), since \( \ell_{\infty} \{ M, \Delta^m, \Lambda, \rho \} \) is a linear space.

Hence \( \ell_{\infty} \{ M, \Delta^m, \Lambda, \rho \} \) is complete. The other results follow in similar way.
PROPOSITION 1. The spaces \( c(B^m, \Lambda, p) \); \( c_0(B^m, \Lambda, p) \) and \( \ell^\infty_\infty(B^m, \Lambda, p) \)
are K-spaces.

Proof follows by the Theorem 3.

THEOREM 4. Let \( M = (M_b) \) be a sequence of Orlicz functions. Then \( c_0(B^m, \Lambda, p) \subset c(B^m, \Lambda, p) \subset \ell^\infty_\infty(B^m, \Lambda, p) \)
and the inclusion is proper.

PROOF. The inclusion \( c_0(B^m, \Lambda, p) \subset c(B^m, \Lambda, p) \) is obvious. We need to show
\( c(B^m, \Lambda, p) \subset \ell^\infty_\infty(B^m, \Lambda, p) \).

Let \( x = (x_k) \in c(B^m, \Lambda, p) \). Then there exists some positive number \( \rho > 0 \) such that,
\[
\lim_{k \to \infty} \left( \left| \lambda_k \left( \sum_{i=1}^k \left| x_i \right| \right)^{\rho} \right| \right) = 0.
\]

Since \( M_b \) is non-decreasing and convex, we have
\[
\left[ M_k \left( \frac{\lambda_k \left( \sum_{i=1}^k \left| x_i \right| \right)}{\rho} \right) \right]^{\rho} \frac{r_k}{D} \leq \left[ M_k \left( \frac{\lambda_k \left( \sum_{i=1}^k \left| x_i \right| \right)}{\rho} \right) \right]^{\rho} \frac{r_k}{D} + D \max \left[ M_k \left( \frac{\lambda_k \left( \sum_{i=1}^k \left| x_i \right| \right)}{\rho} \right) \right] \frac{r_k}{D}
\]
where \( H = \sup_{k} p_k \) and \( D = \max \left( 1, 2^{k+1} \right) \).

Thus \( x \in \ell^\infty_\infty(B^m, \Lambda, p) \). Next to show that the inclusions are strict consider the following example.

Example 1. Let \( p_k = 3 \), for \( k \) even and \( p_k = 4 \), for \( k \) odd. Let \( m \geq 0 \) be given.

Let \( M_b(x) = x^2 \), for all \( x \in [0, \infty) \) and \( \lambda_k = 1 \), for all \( k \in \mathbb{N} \).
Consider the sequence \( x = (k^m, k^m, k^m, \ldots) \).
Then \( x \in c(B^m, \Lambda, p) \), but \( x \) does not belongs to \( c_0(B^m, \Lambda, p) \).

COROLLARY 2. The spaces \( c_0(B^m, \Lambda, p) \) and \( c_0(B^m, \Lambda, p) \) are nowhere dense subsets of \( \ell^\infty_\infty(B^m, \Lambda, p) \). The proof of the result follows from theorem 1. The proofs of the following results are easy, so omitted.

THEOREM 5. Let \( M = (M_b) \) and \( U = (U_b) \) be sequences of Orlicz functions those satisfy the \( \Delta_2 \)-condition, then:

(a) if \( (p_k) \in \ell^\infty_\infty \), then \( Z_0 \) \( (B^m, \Lambda, p) \subset Z_0 \) \( (B^m, \Lambda, p) \);
(b) \( Z_0 \) \( (B^m, \Lambda, p) \cap Z_0 \) \( (U^m, \Lambda, p) \subset Z_0 \) \( (B^m, \Lambda, p) \);
(c) \( Z_0 \) \( (B^m, \Lambda, p) \subset Z_0 \) \( (B^m, \Lambda, p) \); for \( Z = c_0, c_0 \), \( \ell^\infty_\infty \).

RESULT 1. The spaces \( c_0(B^m, \Lambda, p) \), \( c_0(B^m, \Lambda, p) \) and \( \ell^\infty_\infty(B^m, \Lambda, p) \) are not solid, for \( m > 0 \).

Proof of the result follows from the following example.
Example 2. Let \( \lambda_k = 1 \), for all \( k \in \mathbb{N} \); \( M_k(x) = x^2 \), for all \( x \in [0, \infty) \) and \( p_k = 1 \), for \( k \) even and \( p_k = 2 \), for \( k \) odd. Consider a sequence \( (x_k) \) defined by \( x_k = x_k^j \), where \( x_k^j = (k^{-m-1}, k^{-m-1}, \ldots) \), for all \( k \in \mathbb{N} \), then \( (x_k) \in c_0(\mathbb{M}, \Delta^m, \Lambda, p) \).

Now consider \( (\alpha_k) \) defined as \( \alpha_k = (-1)^k \), for all \( k \in \mathbb{N} \).
Thus \( (\alpha_k x_k) \) does not belong to \( c_0(\mathbb{M}, \Delta^m, \Lambda, p) \), for \( m > 0 \).
To show that \( c(\mathbb{M}, \Delta^m, \Lambda, p) \) and \( \ell_\infty(\mathbb{M}, \Delta^m, \Lambda, p) \) are not solid, for \( m > 0 \).
Consider the sequence \( x = (x_k) = (k^m) \) and \( \alpha_k = (-1)^k \), for all \( k \in \mathbb{N} \).

RESULT 2. The spaces \( c_0(\mathbb{M}, \Delta^m, \Lambda, p) \), \( c(\mathbb{M}, \Delta^m, \Lambda, p) \) and \( \ell_\infty(\mathbb{M}, \Delta^m, \Lambda, p) \) are not symmetric, for \( m > 0 \).

PROOF. The spaces are not symmetric follows from the following example.

Example 2. Let \( \lambda_k = k \), for all \( k \in \mathbb{N} \); \( M_k(x) = x^2 \), for all \( x \in [0, \infty) \) and \( p_k = 1 \), for \( k \) odd and \( p_k = 2 \), for \( k \) even. Consider a sequence \( x = (x_k) \) defined by:
\[
(x_k) = (k^2, k^2, k^2, \ldots), \text{ for all } k \in \mathbb{N}.
\]
Then \( x \in c_0(\mathbb{M}, \Delta^m, \Lambda, p) \), for \( m = 1 \).
Let the rearrangement \( (y_k) \) of \( (x_k) \) be,
\[
(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, \ldots).
\]
Then \( (y_k) \) does not belong to \( Z(\mathbb{M}, \Delta^m, \Lambda, p) \), for \( m > 0 \), where \( Z = c_0, \ell_\infty \).
Hence the result follows. Proofs of the following results follows from the lemmas listed in the section 2.

PROPOSITION 2. Let \( h = \inf_k p_k \), \( H = \sup_k p_k \). Then the following conditions are equivalent:
(i) \( H < \infty \) and \( h > 0 \);
(ii) \( c_0(\mathbb{M}, \Delta^m, \Lambda, p) \equiv c_0(\mathbb{M}, \Delta^m, \Lambda, p) \);
(iii) \( \ell_\infty(\mathbb{M}, \Delta^m, \Lambda, p) \equiv \ell_\infty(\mathbb{M}, \Delta^m, \Lambda, p) \).

PROPOSITION 3. Let \( p, q \) be two sequences of strictly positive numbers. Then \( c_0(\mathbb{M}, \Delta^m, \Lambda, p) \equiv \ell_\infty(\mathbb{M}, \Delta^m, \Lambda, q) \) if and only if there exists a sequence \( u = (u_k) \) of strictly positive numbers such that the equations \((1)\) and \((2)\) holds.

PROPOSITION 4. Let the sequence \( v = (v_k) = (q_k^p, p_k^{p_k^{-1}}) \). Then \( c_0(\mathbb{M}, \Delta^m, \Lambda, p) \equiv c_0(\mathbb{M}, \Delta^m, \Lambda, q)(v) \) if and only if the equations \((3)\) and \((4)\) holds.
PROPOSITION 5. Let the sequence \( v = (v_k) = \left( q_k^{-\frac{1}{s}} p_k^{-\frac{1}{r}} \right) \). Then \( \lim_{k \to \infty} \frac{1}{p_k} - \frac{1}{q_k} = 0 \) implies \( c_0(M, \Delta^\infty, \Lambda, p) \cong c_0(M, \Delta^\infty, \Lambda, q)(v) \).

PROPOSITION 6. Let \( g_k = \frac{p_k}{q_k} \), for every \( k \in \mathbb{N} \). Let \( (g_k) \) and \( (g_k^{-\frac{1}{s}}) \) both be in \( \ell_\infty \). Then \( \ell_\infty(M, \Delta^\infty, \Lambda, p) \cong \ell_\infty(M, \Delta^\infty, \Lambda, q, g) \).

PROPOSITION 7. Let \( q \) be a bounded sequence of strictly positive numbers. Then \( \ell_\infty(M, \Delta^\infty, \Lambda, p) \subseteq \ell_\infty(M, \Delta^\infty, \Lambda, q) \) if and only if the equation (5) hold.

PROPOSITION 8. Let \( q \) be a bounded sequence of strictly positive numbers and \( c_0(M, \Delta^\infty, \Lambda, p) \cong c_0(M, \Delta^\infty, \Lambda, q) \), then \( c_0(M, \Delta^\infty, \Lambda, p) \cong c_0(M, \Delta^\infty, \Lambda, q) \).

REFERENCES


SUGGESTED CITATION


http://www.akamaiuniversity.us/PJST.htm

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