A Six-Step Scheme for the Solution of Fourth Order Ordinary Differential Equations.

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ABSTRACT

A linear multistep method for solving fourth order initial value problems of ordinary differential equations is presented in this paper. The approach of collocation approximation is adopted in the derivation of the scheme and then the scheme is applied as simultaneous integrator to special fourth order initial value problem of ordinary differential equations. This implementation strategy is more accurate and efficient than those given when the same scheme is applied over overlapping intervals in predictor-corrector mode. Furthermore, the new block method possesses the desirable feature of Runge-Kutta method of being self-starting and eliminates the use predictor- corrector method. The method is symmetric and zero-stable. Numerical examples are also given.

(Keywords: linear multistep methods, LMMs, p-stability, zero-stability, third order, IVPs, odes, interval of periodicity, predictor-corrector)

INTRODUCTION

The special fourth order initial value problems (IVPs) given as:

$$y^{iv} = f(x, y); y(a) = y_0, y'(a) = \eta_0, y''(a) = \eta_1, y'' = \eta_2 \quad \textbf{(1)}$$

is considered. We assume that the numerical solution is required on a given set of mesh.

$$\prod = \{x_n / x_n = a + nh, h = x_{n+1} - x_n, n = 0,1,......N\}$$

where N = b-a/h.

Recent research in this area includes Awoyemi and Idowu (2005) a class of hybrid collocation method for general third order of ordinary differential equations. Awoyemi (2003) developed a p-stable linear multistep method for general third order initial value problems of ordinary differential equations but the implementation strategy is in predictor-corrector mode. Like other linear multistep methods and other standard methods, are usually applied to the initial value problems as a single formula but the drawbacks of the methods are well known. Firstly they are not self-starting; secondly, they advance the numerical integration of the ordinary differential equations in one-step at a time, which leads to overlapping of the piecewise polynomials solution model (Yusuph, 2004). Moreover, the overlapping creates a disadvantage because the numerical model fails to represent the solution uniquely elsewhere than the mesh-points. For boundary value problems, this is an important criticism of the linear multistep methods in favor of the finite element methods (Jennings, 1997).

Jator (2007) presented a class of initial value methods for the direct solution of second order initial value problems. In his work, LMMs with continuous coefficients were obtained and applied as simultaneous numerical integrators to $y'' = f(x, y, y')$. The implementation strategy is more efficient than those given in Awoyemi (1999) which are applied over overlapping intervals in predictor-corrector mode. Vigo-Aguiar & Ramos (2006) constructed a variable step-size implementation of multistep Methods for, $y'' = f(x, y, y')$. Yusuph and Onumanyi (2005) derived new multiple FDMS through multistep collocation for $y'' = f(x, y)$. Fatokun and Onumanyi (2007) also derived second and fourth order two-step discrete finite difference methods by collocation for the first approximation and combined them with the Numerov method for a direct application to general second order initial value problem of ODEs.

This paper therefore proposes 6-step block scheme for the solution of special fourth order ordinary differential equations which eliminates the use of predictors by providing sufficiently accurate simultaneous difference equations from a single continuous formula and its derivative.

METHODOLOGY

A power series of a single variable x in the form

$$P(x)=\sum_{j=0}^{\infty}a_j x^j \qquad \textbf{(2)}$$

is used as the basis or trial function, to produce the approximate solution as,

$$y(x)=\sum_{j=0}^{k+2}a_j x^j$$

$$a_j \in R,\ j=0(1)k+3, y\in C^m(a,b)\subset P(x). \qquad \textbf{(3)}$$

Assuming an approximate solution to (1) in the form of (3) whose high derivatives are:

$$y'(x)=\sum_{j=0}^{k+2}j(j-1)a_j x^{j-1} \qquad \textbf{(4)}$$

$$y''(x)=\sum_{j=0}^{k+2}j(j-1)(j-2)a_j x^{j-2} \qquad \textbf{(5)}$$

$$y'''(x)=\sum_{j=0}^{k+2}j(j-1)(j-2)(j-3)a_j x^{j-3} \qquad \textbf{(6)}$$

$$y^{iv}(x)=\sum_{j=0}^{k+2}j(j-1)(j-2)(j-3)(j-4)a_j x^{j-4} \qquad \textbf{(7)}$$

From Equations (2) and (7):

$$\sum_{j=0}^{k+2}j(j-1)(j-2)(j-3)(j-4)a_j x^{j-4}=f(x,y) \qquad \textbf{(8)}$$

where a_j are the parameters to be determined. Collocating (8) at the mesh–points $x=x_{n+j}$, j=0(2)k, and interpolating (3) at $x=x_{n+j}$

j = 2, 3, 4 and 5 yield a system of equations.

Putting in the matrix equation form and then solved to obtain the values of parameters a_j's , j = 0, 1, .. which is substituted in (3) yields, after some manipulation, the new continuous method expressed in the form:

$$y_6(x)=\sum_{j=0}^{k-1}a_j(x)y_{n+j}+\sum_{j=0}^{k}\beta_{2j}(x)f_{n+2j} \qquad \textbf{(9)}$$

DERIVATION OF FIRST- BLOCK OF 6-STEP BLOCK METHOD

The first-block of 6-step method for the fourth order initial value problem (ivp) designated by Equation (1) can be expressed by the following matrix difference equation:

$$A^{(0)}.\,y_q = A^{(1)}.y_{q-1} + h^4B^{(0)}.F_q + BF_{q-1} \qquad \textbf{(10)}$$

From (9) the coefficients $\alpha(x)\ and\ \beta(x)$ are thus:

$$a_2=\frac{1}{6}(-t^3+9t^2-26t+24)$$

$$a_3=\frac{1}{6}(3t^3-24t^2+57t-36)$$

$$a_4=\frac{1}{6}(-3t^3+21t^2-42t+24)$$

$$a_5=\frac{1}{6}(t^3-6t^2+11t-6)$$

$$\beta_0=\frac{h^4}{40320}\{-t^7+21t^6-161t^5+525t^4-434t^3-1386t^2+3116t-1680)$$

$$\beta_2=\frac{h^4}{40320}\{3t^7-49t^6+147t^5+1575t^4-13258t^3+38794t^2-49052t+21840) \qquad \textbf{(11)}$$

$$\beta_4=\frac{h^4}{40320}\{-3t^7+35t^6+21t^5-525t^4-3262t^3+23170t^2-41276t+21840)$$

$$\beta_6=\frac{h^4}{40320}\{t^7-7t^6-7t^5+105t^4+154t^3-1778t^2+3212t-1680)$$

$$y_{n+6} - 4y_{n+5} + 6y_{n+4} - 4y_{n+3} + y_{n+2} = \frac{h^4}{24}\{ f_n + 22 f_{n+2} + f_{n+6})$$ **(12)**

The order $P = 6$ i.e $C_{p+2} = -0.0431$

Region of absolute stability x (θ)=(0.16) where:

$f_n = f(x_n, y_n)$, $f_{n+1} = f(x_{n+1}, y_{n+1})$, $f_{n+2} = f(x_{n+2}, y_{n+2})$,

$f_{n+3} = f(x_{n+3}, y_{n+3})$, $f_{n+4} = f(x_{n+4}, y_{n+4})$, $f_{n+5} = f(x_{n+5}, y_{n+5})$,

$f_{n+6} = f(x_{n+6}, y_{n+6})$

The first, second and third derivatives of (11) are found, noting that:

$$\frac{dt}{dx} = \frac{1}{h} \quad and \quad \frac{d^2t}{dx^2} = \frac{1}{h^2}, \ \frac{d^3t}{dx^3} = \frac{1}{h^3}$$ **(13)**

APPLICATION OF 6-STEP BLOCK METHOD

1. Evaluation of equation (11) at $x = x_n$, $x = x_{n+1}$ and $x = x_{n+6}$ yield respectively the three integrators below.

$$y_{n+6} - 4y_{n+5} + 6y_{n+4} - 4y_{n+3} + y_{n+2} = \frac{h^4}{24}(f_{n+2} + 22 f_{n+4} + f_{n+6})$$

$$with \ \text{order } P = 6 \text{ i.e } C_{p+2} = -0.0431$$

$$y_{n+5} - 4y_{n+4} + 6y_{n+3} - 4y_{n+2} + y_{n+1} = \frac{h^4}{24}(- f_n + 13 f_{n+2} + 13 f_{n+4} - f_{n+6})$$

$$\text{order } P = 6 \text{ i.e } C_{p+2} = -0.025$$

$$4y_{n+5} - 15y_{n+4} + 20y_{n+3} - 10y_{n+2} + y_n = \frac{h^4}{24}(-3 f_n + 74 f_{n+2} + 53 f_{n+4} - 4 f_{n+6})$$

$$\text{order } P = 6 \text{ i.e } C_{p+2} = -0.06798$$

(14)

2. Improving the block method by considering additional equation arising from the first and second derivative functions;

$$\frac{du(x)}{dx} = Z(x), \frac{du(a)}{dx} = Z_0$$

$$\frac{d^2u(x)}{dx^2} = Z'(x), \frac{d^2u(a)}{dx^2} = Z_0'$$

$$\frac{d^3u(x)}{dx^3} = Z''(x), \frac{d^3u(a)}{dx^3} = Z_0''$$

(15)

Using the first and second conditions in (1) on (15) to obtain:

$$hZ_0 - \frac{13}{3}y_{n+5} + \frac{31}{2}y_{n+4} + \frac{38}{2}y_{n+3} + \frac{47}{6}y_{n+2} = \frac{h^4}{40320}(1548 f_n - 171664 f_{n+2} - 95428 f_{n+4} + 6824 f_{n+6})$$

$$h^2 Z' + 3y_{n+5} - 10y_{n+4} + 11y_{n+3} - 4y_{n+2} + y_{n+1} = \frac{h^4}{40320}(10024 f_n - 171500 f_{n+2} + 60368 f_{n+4} - 3332 f_{n+6})$$

$$h^3 Z'' - y_{n+5} + 3y_{n+4} - 3y_{n+3} + y_{n+2} = \frac{h^4}{40320}(-27594 f_n - 102018 f_{n+2} - 10542 f_{n+4} - 966 f_{n+6}) \quad \textbf{(16)}$$

The above discrete schemes are uniformly of order 6 and of close accuracy could be used to start the initial value problem integration over (x_0, x_6), n = 0, 6…. Combining (14) and (16) give the first block. There are two ways to advance the integration process after the first sub-interval {x_o, x_5}, where the block method simultaneously provides values y_1, y_2 … without recourse to any other one-step method to provide them. This is an improvement over the use of block method, which is not self-starting.

We shall now discuss two options to move the integration process forward after the first sub-interval. First, we could proceed by using the first block 3-point in Fatunla (1994) i.e block method over sub-intervals that do over-lap (x_4 , x_6}, {x_5, x_7}…., {X_{N-4}, X_N}.

The second options is to proceed by explicitly obtaining initial conditions at x_{n+6}, n= 0, 6,…N-6, using the computed values $U(x_{n+6}) = y_{n+6}$ and $Z(x_{n+6})=Z_{n+6}$ over sub-intervals (x_o, x_6} {x_6, x_{12}}….{x_{N-6}, x_N} which do not over-lap.

NUMERICAL EXPERIMENTS

This section deals with the implementation of the algorithm proposed for fourth order initial value problems.

1. $y^{iv} = \frac{Q(x)}{EI}; 0 \le x \le 1$

$$y(0) = 0, \; y(0) = \frac{\phi(x)}{48}, \; y''(0) = 0, \; y'''(0) = \frac{-3\phi(x)}{8}$$

$$\phi(x) = E = I = 1$$

Theoretical solution

$$y(x) = \frac{\phi(x)}{48}(2x^4 - 3x^2 + x)$$

Table 1: Absolute Errors for the Method.

Theoretical/Approximate Solutions, h=0.1			
X	**Exact solution y(x)**	**6-stepBlockMethod y-computed**	**Errors**
0.1	0.001462500000	0.002020835	5.58335159E-04
0.2	0.001733333333	0.003666721	1.93338810E-03
0.3	0.000962500000	0.004562906	3.60040589E-03
0.4	-0.000600000000	0.00433499	4.93498971E-03
0.5	-0.002604166667	0.002609012	5.21317830E-03
0.6	-0.004600000000	-0.000988581	3.61141891E-03
0.7	-0.006037500000	-0.006831112	7.93611900E-04
0.8	-0.006266666667	-0.015292039	9.02537270E-03
0.9	-0.004537500000	-0.02674555	2.22080499E-02
1	0.000000000000	-0.02674555	2.67455499E-02

Table 2: Accuracy of 6-Step Block Method of Order 6, H=0.1.

Theoretical/Approximate Solutions, h=0.1			
X	**Exact solution y(x)**	**6-stepBlockMethod y-computed**	**Errors**
0.1	0.100000083	0.100000084	1.66666667E-10
0.2	0.200002667	0.200002667	3.33333305E-10
0.3	0.30002025	0.300020251	5.99999994E-10
0.4	0.400085333	0.400085334	7.66666675E-10
0.5	0.500260417	0.500260418	9.33333300E-10
0.6	0.600648	0.600648001	1.10000009E-09
0.7	0.701400583	0.701400585	1.27166666E-09
0.8	0.802730667	0.802730668	1.45333334E-09
0.9	0.90492075	0.904920752	1.64999991E-09
1	1.008333333	1.008333335	1.87666660E-09
1.1	1.113420917	1.113420919	2.13333329E-09
1.2	1.220736	1.220736002	2.42999998E-09

2. $y^{iv} = x$

$$y(0)=0, \quad y'(0)=1, \quad y''(0)=y'''(0)=0$$

Theoretical solution is

$$y(x) = \frac{x^5}{120} + x$$

CONCLUSION

The 6-step block method proposed for special fourth order initial value problems eliminates the use of predictor-corrector methods and it is also more accurate and faster than the conventional (step-step) integration procedures.

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SUGGESTED CITATION

Olabode, B.T. 2009. "A Six-Step Scheme for the Solution of Fourth Order Ordinary Differential Equations". *Pacific Journal of Science and Technology.* 10(1):143-148.

Pacific Journal of Science and Technology